Appendix

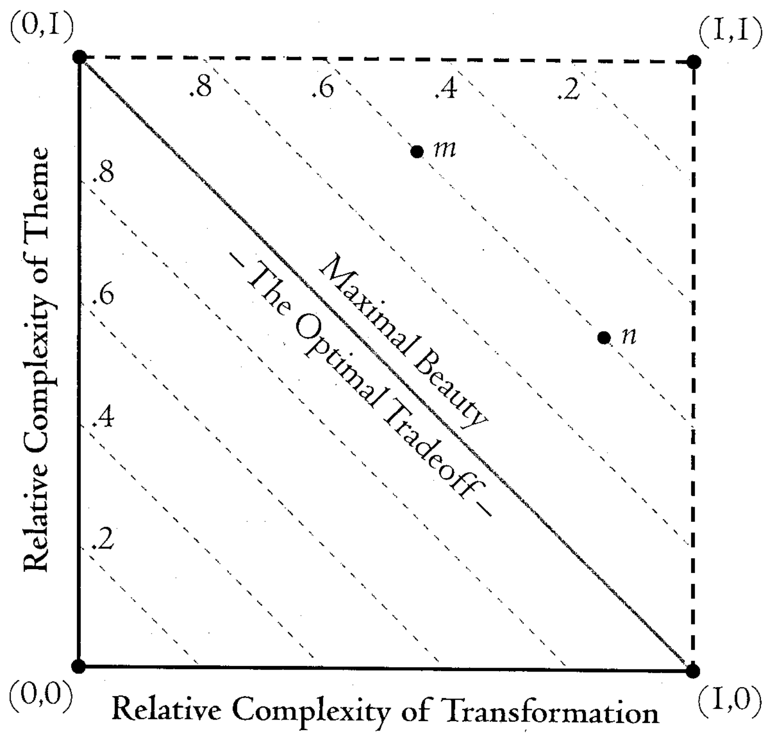
The Fundamental Theorem of Beauty-in-Design

In this appendix, I integrate the ideas about aesthetic beauty set forth above into a formal theory of the measurement of relative beauty-in-design. More specifically, I seek to identify a function Z whose values range from 0 to 1 (with 0 representing "least beautiful" and 1 "most beautiful") that can be used both to predict and to explain the relative beauty of any object or musical composition of the class considered here. I present this appendix partly because the serious study of aesthetics has always been mathematical, since the time of Pythagoras, and partly in hopes that students reading this essay will be encouraged to pursue the field with renewed enthusiasm—unencumbered by modernist prejudices that the subject of beauty is merely "subjective." For it most definitely is not.

Consider, as in the graph at right, a pair of x and y axes, where x denotes on a 0,1 scale the degree of relative complexity of the *theme* of a work of art, and y denotes on a 0,1 scale the degree of relative complexity of the *transformations* of this theme needed to generate the remainder of the object's design. Any object can be represented by two such coordinates, for reasons explained in the main text.[1]

The unit square induced by this representation and graphed at right has four vertices: the bottom left vertex (0,0) corresponds to an object characterized by a maximally simple theme and transformation (and thus maximally boring), whereas the upper right vertex (1,1) represents a maximally complex object. The remaining two vertices are the endpoints (0,1) and (1,0) of the principal downward diagonal that bisects the square into two equal regions. In light of the discussion in the main text, the set of all points on this negatively sloped 45-degree line represents the equivalence class of *maximally beautiful designs*, insofar as these points represent all designs capturing the *optimal tradeoffs* between the relative complexity of both themes and transformations thereof.

Representation of Aesthetic Beauty



Given these assumptions, I now seek a function $Z = f(x,y)$ with values lying between 0 and 1 that will rank any and every object in terms of its *relative beauty-in-design*, in particular, a function that satisfies the following three axioms (desiderata) that summarize what the present theory is all about at a highly abstract level. I also ask whether such a function, if it exists, is the *only* function consistent with the three axioms. The fundamental theorem below answers both questions in the affirmative.

Axiom 1: Linearity: Z will be linear in x and y. [Strictly, it is piecewise linear with a break at the main downward diagonal along which Z achieves its highest values, as will be seen below.] **Rationale:** This axiom requires that the impact-on-beauty of changes in y be independent of the value of **x**, and vice versa. Each variable thus enters the analysis separately. Linearity also requires that the impact on the value of Z of unit changes in the value of either **x** or **y** be the same everywhere on the domain of Z. Both these implications of linearity can be viewed as *non-discrimination symmetries* requiring first that **x** and **y** can vary independently of one another, and second that the theory applies everywhere on the **X•Y** domain with the same force.

Axiom 2: Invariance under Permutation: Z will assign equal values to any two points (**x,y**) and (**x',y'**) in the **X•Y** domain that are equivalent under the operation of permutation. This is the case of the two points **m** and **n** in the graph where the coordinates are respectively (.6, .8) and (.8, .6). **Rationale:** This axiom asserts that the value achieved by Z cannot be biased by the *type* of complexity at hand. Both complexity of theme and complexity of transformation count equally in the beauty ranking.

Axiom 3: Strict Monotonicity: For any given point (**x,y**) off of the main diagonal, the value of Z must increase if the distance of **x** from the main downward diagonal is reduced—holding **y** constant; and vice versa for any reduction in the distance of **y** from the same diagonal. **Rationale:** The farther we move away from the two "ugly" corners (0,0) and (1,1), the better.

THEOREM: There exists a unique measurement-of-beauty function Z mapping any point on the unit square into the closed interval [0,1] of the real line that satisfies these three simple axioms. This function is:

(1.A) $Z = x + y$ when the sum $x + y$ is less than or equal to unity, and

(1.B) $Z = (1 - x) + (1 - y)$ when the sum $x + y$ exceeds unity, as will be true of all points lying in the region of the graph above the main downward diagonal.

PROOF: It can be checked by inspection that the proposed function satisfies each of the three axioms. The proof of uniqueness is trivial given Axiom 1 in conjunction with the restriction of the domains of Z, **x**, and **y** respectively to the unit interval of the real line. In particular, no affine transformation of (1) will be admissible. ▼

NOTE: The function (1) essentially defines a family of iso-beauty lines parallel to the main downward diagonal. These lines have beauty-values that descend from 1 to 0 as their distance (in either direction) from the main diagonal increases. All points on any given such line will have the same beauty score, as indicated in the graph, and all points on the main diagonal will have a score of 1. Now, this extremely simple function (1) *could* be interpreted as a linear approximation to some more general quadratic form corresponding to a more complex and better theory of beauty, e.g., $Z = Q(x,y)$. However, while I originally believed this to be the case, I no longer do. Applications of the theory for two decades strongly suggest that the current linear version proposed above is sufficient.

Why might a simple linear theory suffice? As is so true in much serious mathematics, fundamental theorems end up being surprisingly simple, provided that all the heavy lifting has been done by the underlying *definitions*. The case in hand offers a good example of this maxim since the two definitions of relative complexity introduced in the theory are extremely sophisticated.[2] Hopefully, this new result proffers a good start to a generalized mathematical analysis of beauty-in-design, one similar in spirit but much deeper in content than that proposed by G. D. Birkhoff over seventy years ago.

NOTES

1. Representing any object as a point in this particular space requires that we take the "degree of complexity" of its theme and of its transformations, as defined in the main text, and rescale each as a number lying between 0 and 1. In principle, this should not be difficult, at least to a first approximation.

2. Analogously, in the economics of uncertainty, expected utility is linear in utility. Yet utility itself is a very complex concept that had to be developed before the deceptively simple expected utility theorem could be proved by John von Neumann in 1947. In pure mathematics, the fundamental theorem of the calculus in **n**-dimensions can be proven in the space of half a page once the concepts of manifolds, exterior differential forms, and wedge products have been introduced. Prior to the discovery of these abstract concepts, the same theorem required a fifteen-page proof. Perhaps most impressively, in general relativity theory, all "motion" takes the form of a simple straight line in curved space-time known as a geodesic. But this only became clear once Einstein demonstrated that space-time itself is "curved," a concept that required the new mathematics of differential geometry.

The Truth about Beauty

HORACE WOOD BROCK

Beauty is truth, truth is beauty—that is all ye know on earth, and all ye need to know.
John Keats, *"Ode on a Grecian Urn"*

My interests as a collector have primarily lain in what are called (inappropriately, to my mind) "decorative arts," mainly the decorative arts of France and England in the golden age of 1675–1820, with a subsidiary interest in Old Master drawings from 1500 to 1820. If there is anything distinctive about this collection, it is that it was formed in accordance with a theory that I developed about what makes objects "beautiful"—the nature of *truth about beauty,* in the Keatsian sense. This theory is summarized in the second part of this essay, but first I would like to address some questions I have been asked about how my collection was formed over three decades.

I *Creation and Motivation*

Looking back, I can trace two developments in my life that most directly shaped my collecting interests. First, during my teenage years, I often visited my maternal uncle in London. He had a house in Mayfair filled with good eighteenth-century French and English works of art. He would explain them to me and I instinctively appreciated them, never thinking that I would one day collect such items on my own.

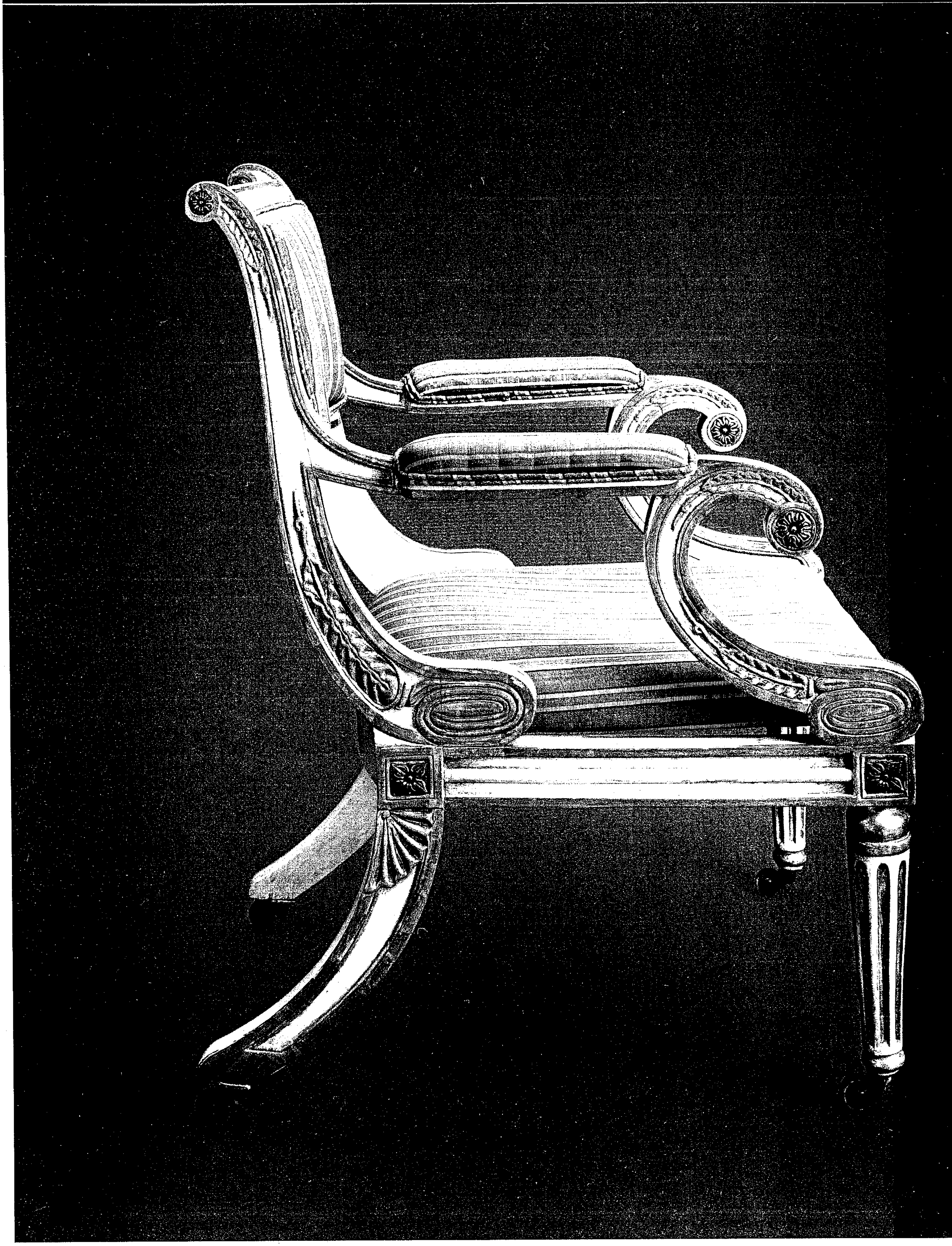
Second, I was a special student in classical music (piano) at the New England Conservatory while at boarding school and then at Harvard. My serious involvement in music, along with a commitment to the study of Greek and Latin and to European history and languages, played an ancillary role in stimulating my subsequent love of European arts of all kinds. As I matured, I came to appreciate the natural affinities that exist between classical music, architecture, decorative arts, and fine arts. For example, the term "Rococo" to me evokes equally the Amalienburg Pavilion at the Nymphenburg Palace in Munich, the Meissonier-style candlesticks shown in this catalogue (cat. no. 5), the minuets of Mozart, and the paintings of Boucher (cat. no. 94). Analogously, "Neoclassical" evokes equally Beethoven's *Emperor Concerto,* the paintings of Jacques-Louis David, the "Vestal Virgin" clock attributed to Pierre-Philippe Thomire (cat. no. 44), and the Great Hall of Syon House on the river Thames. The underlying message in each of these cases is one and the same.

My motivation for collecting is best summarized by what I tell friends who ask why I have acquired and live with such objects: "For the same reason you hear the music of Bach and Beethoven when you enter this house: the quest for beauty in the refuge of my home." Nothing else ever motivated me. As for the prospects of financial gain, I realized as an economist that attempting to "beat the market" or find bargains would prove ill-fated. How could a novice exploit the same arbitrage opportunities that established dealers can, with their years in the business? In fact, the only time I did sell a part of my collection, I actually lost a bit of money!

Just as the prospect of financial gain never motivated me, neither did the quest for social status. The stark reality is that the kinds of things I have loved have become ever more unfashionable since I began collecting thirty years ago. Bluntly, beauty per se is out, and "interesting" shock-art is in. And as for my love of elegance, God forbid! This is today's form of "the love that dare not speak its name." As a result of such changing tastes, there has never been a better time to acquire works of great beauty and elegance. Drawing upon my own experience, my *strategic* advice to young collectors is to acquire objects solely because they find them truly beautiful and life-enhancing, and not for their "investment potential." My tactical advice is to heed the cautionary note often attributed to the legendary collector and banker J. Pierpont Morgan: "The only true bargain is quality."

I have bought almost exclusively from dealers rather than from the sale room. This tends to be more expensive, given the large commissions dealers charge, but the education one can acquire from dealers, who have an incentive to discern and acquire "the best," is priceless. The many hours I spent with such legendary London dealers as George Levy, Martin Levy, Robin Kern, Frank Berendt, and the Hill brothers, and with several notable dealers in Paris, were invaluable in this regard. I am also particularly indebted to the expertise and patience of Theodore Dell, who helped me build my collection of French decorative arts. Watching these individuals work over the years and learning from

Henry Holland chair (see cat. no. 72)



out, as in the case of the Holland chair, or in Mozart's celebrated variations on the theme of "Twinkle, Twinkle, Little Star," but this is not always the case.

"Relative complexity"

Now let us consider the *relative complexity* of both the themes and the transformations. Both concepts of relative complexity can be represented in simple mathematical terms. In the case of the theme of the Holland chair, an appropriate measure of the complexity can be objectively given as the degree of the polynomial equation representing the graph of the arm's S-curve. This equation is cubic, so it is of degree 3. In the case of a kitchen chair with splats across the back, the theme is a straight line segment. This can be represented as a linear equation of degree 1. The *higher* the degree of this equation, the more *complex* the theme is said to be.[1]

The relative complexity of each transformation can be defined analogously, although the mathematics is more complex. In the case of the Holland chair, both the 90-degree rotation and the mirror-image reflection can be considered simple operators, whereas convolution (twisting) is complex. The more work required to transform a given theme, the more *complex* the transformations are said to be.[2] This is true regardless of whether the transformations involved are static, as in furniture or architecture, or dynamic, as in music. (In this context, it is worth recalling Schelling's description of architecture as "frozen music.")

Armed with these concepts, we can offer an analytical account of the degree of satisfaction that we derive from the beauty of an object's design in terms of its intrinsic structural properties—in particular, its symmetries, coherence, and harmony. To understand this point, first note that maximal symmetry and/or harmony can be seen to correspond to the combination of a simple theme and simple transformations thereof: for example, a simple kitchen chair with equally spaced linear splats constituting its back, or a simple Gregorian chant in music. In the opposite extreme, maximal disorder and/or incoherence can be seen to correspond to the combination of a complex theme and a complex transformation of this theme. Much Victorian furniture and twentieth-century atonal music is highly complex in this regard.

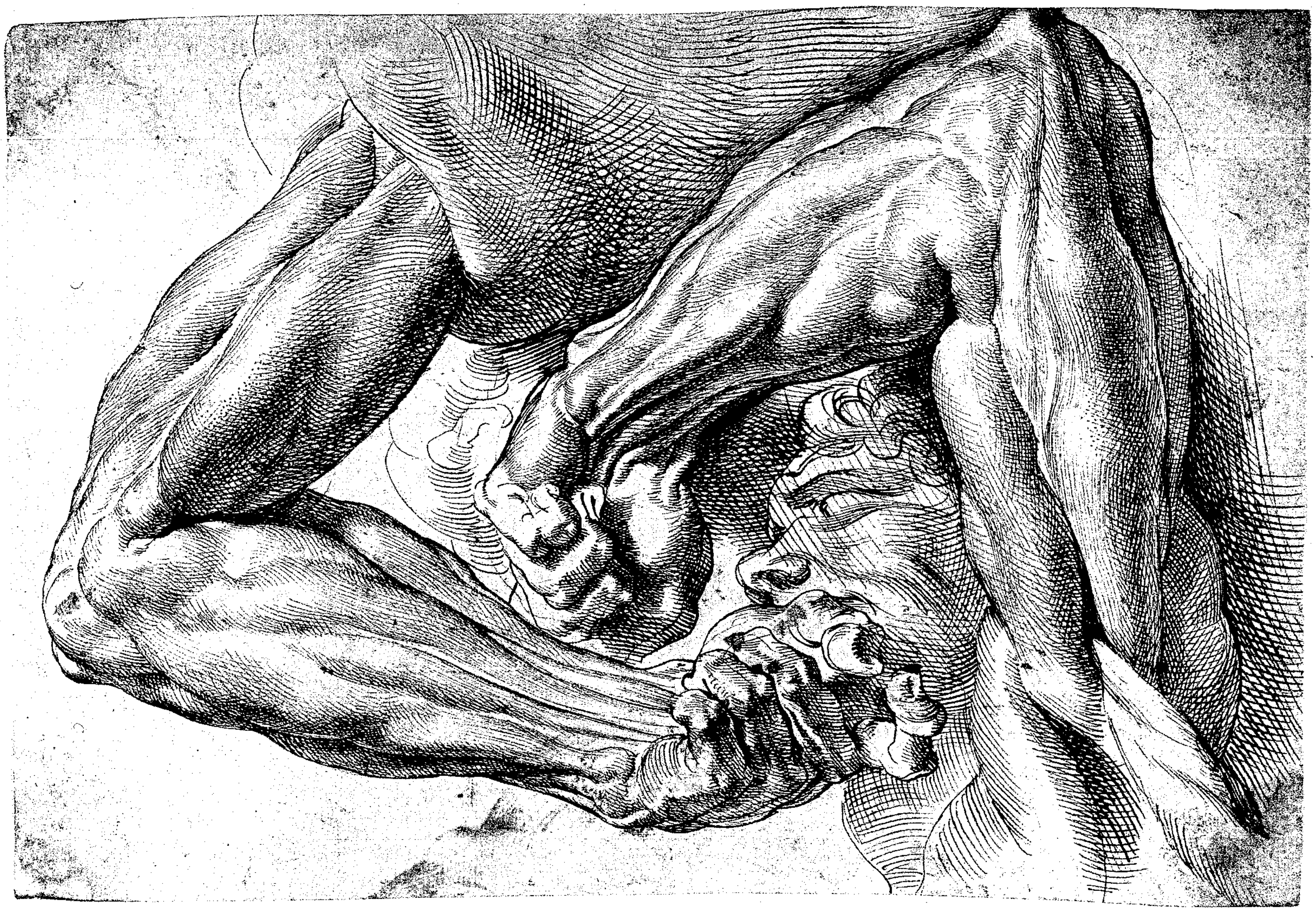
Where does aesthetic satisfaction lie? My fundamental thesis here is that the highest degree of satisfaction typically results when the right balance is achieved between order and disorder—that is, when an "optimal tradeoff" is struck between the two different dimensions of relative complexity that we have just identified. If the theme is simple, then we are most satisfied when its echoes are complex ("interesting"), and vice versa.

This account makes it possible to clarify, and indeed to quantify, one of the deepest principles of aesthetics: people apprehending works of art tend to be bored if there is too much simplicity (the kitchen chair, certain Gregorian chants), and overwhelmed if there is too much complexity (pastiche Victorian furniture, much twentieth-century classical music). People wish to be optimally challenged by experiencing the proper balance of total complexity. Another way to state this is that there is an optimal amount of symmetry (or asymmetry) that is apparently a necessary condition for a work of art to be found "pleasing." This principle runs throughout aesthetic theory from the Greeks on down.

It follows that an object possessing a very simple theme but sufficiently complex transformations can be *as beautiful* (pleasing) as an object with a complex theme but a simple set of transformations. Moreover, either of these two extreme cases can be deemed *as beautiful* as an in-between case, such as the Henry Holland chair with its relatively complex theme and relatively simple transformations. An entire family of completely disparate art objects can thus be deemed equivalently beautiful at an abstract level. Conversely, objects that exhibit either *too much* aggregate complexity (of theme *and* transformation), or *too little* complexity, are generally experienced as ugly—regardless of genre.

The appendix below presents for the interested reader a formal mathematical summary of this theory. Additionally, the fundamental theorem demonstrates that there exists a unique quantitative measure of relative beauty that satisfies the three simple axioms described in the appendix.

Two final points should be made about this theory of aesthetic beauty as it applies to the decorative arts. First, it can be shown to *generalize* most previous theories of aesthetic success in design. Specifically, it subsumes the myriad relationships revolving around the celebrated "golden number" 1.618. For centuries, this algebraic number has been used to account for the aesthetic appeal of such diverse entities as the Pythagorean golden rectangle, the celebrated "golden section," the structure of the human body (as observed by Leonardo), the role of Fibonacci series, the proportions of the ideal Gothic cathedral, the arresting spiral of the nautilus shell and the pinecone, and so on.[3] Yet these previous theories failed to identify the role of relative complexity in aesthetics, and it is this that generates the power and scope of the present, new theory. By subsuming these earlier theories as special cases, while going beyond them, the present theory should make it possible to unify much of aesthetic theory in a new, deeper, and more satisfactory manner than has hitherto been possible.



1.
Peter Paul Rubens
Flemish, 1577–1640
A Sheet of Anatomical Studies,
1600–1605
Pen and brown ink

Second, this new account of beauty is fully context-dependent, or "relativistic" in mathematical terms. To see this, suppose we substitute a straight line arm for the S-shaped arm of the Holland chair. We obtain the following paradox: while a straight arm is much simpler than an S-curved arm in isolation, the chair that results nonetheless becomes much more complicated—so complicated that most people will deem it ugly. This is because too much "aesthetic energy" must now be expended in mentally transforming the straight line of the chair arms into its S-shaped back, for the two design lines are essentially unrelated. Thus, what is simple in one context is not at all simple in another.[4]

Extension of the theory to fine arts, sculpture, and architecture

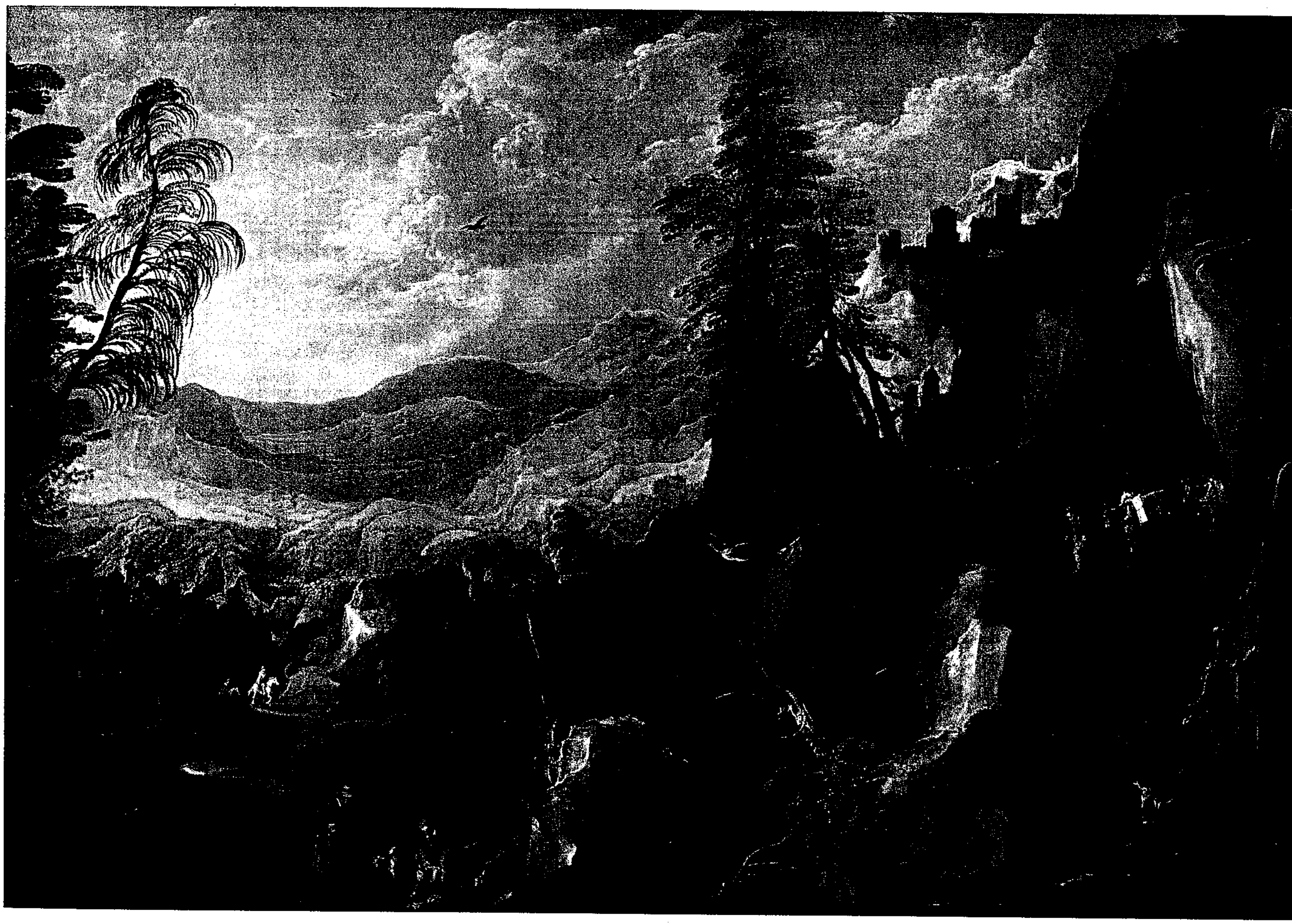
The same theory of beauty has implications for the drawings in my collection as well, and for other two-dimensional works, particularly in their structural or compositional aspects. For example, let us look at the improbable symmetries of the human arms in Rubens's *Sheet of Anatomical Studies* (cat. no. 1). Because of these symmetries, the drawing can be hung either vertically or horizontally with equal effect. This is true of none of the other dozen or so drawings in the album from which this example came, and for this reason, the present drawing is perhaps the most satisfying and interesting of the lot. Additionally, consider the use of dominant diagonals in artistic compositions from the time of the Renaissance (such as the powerful drawing by Monti [cat. no. 2]). Such diagonals add an optimal asymmetry that energizes the entire composition. The same is true of the bold diagonal in the huge landscape painting by Joos de Momper (cat. no. 3).

Interestingly, we see a similar phenomenon at work in some of the most beloved paintings in existence: Van Gogh's *The Starry Night* and *The Olive Trees* (both 1889; Museum of Modern Art, New York) and Munch's *The Scream* (1893; National Gallery of Norway). In all three cases, the theme is highly complex (e.g., the spiral design of Van Gogh's stars), yet the transformations are simple (note that the representation of the clouds and the cypress trees in *The Starry Night* closely resembles the spiral shape of the stars). This is aesthetic coherence and beauty at its high point.

We can use this approach to understand aesthetic success in sculpture and architecture as well. The Greek sculptor Polykleitus reputedly introduced the nude male athlete with a broken stride and a correspondingly bent leg. Contrast this optimally asymmetrical form with earlier, bilaterally symmetric representations of the human body: how boring they are by comparison! Their quality of execution and intricacy of detail might be exquisite, as in the case of beautiful Hindu Shivas, but from the standpoint of overall design, their excessive symmetry makes them less attractive.

2. **Francesco Monti.** Italian, 1685–1768. *Study of a Man Holding a Pole.* Black chalk and charcoal

As for architecture, successful examples include such diverse landmarks as the Chrysler Building in New York, the Bank of China Building in Hong Kong, the Sydney Opera House, the Transamerica Pyramid in San Francisco, the Taj Mahal, the sail-like extension of the lakeside Milwaukee Art Museum, 30 St Mary Axe ("the Gherkin") in London, and the idealized Georgian country house with its central black and symmetrical adjoining wings.[5] Conversely, this theory explains the aesthetic failure of



3.
Joos de Momper
Flemish, 1564–1635
Mountain Landscape with Travelers, 1620s
Oil on canvas

those mind-numbing, excessively symmetrical "cookie-cutter" office buildings and row houses that plague American suburbs.

In short, people respond to the appeal of successful design across all categories of art. The present theory of aesthetic beauty helps explain exactly what it is they are responding to. To put it plainly, viewers seek organicity and "eurythmic" aesthetic coherence, and they eschew both timidity and confusion; they want to be optimally challenged but neither bored nor, conversely, overwhelmed. I hope that looking at works of art in terms of relative complexity will make it possible to better understand and articulate this reality, and to better guide the enterprise of good design in the future.

A final note

I first developed this theory in a series of some forty letters written to the late George Levy in 1981. Thereafter, a compact mathematical summary of the essential points of the theory was published in my essay "Game Theory, Symmetry, and Scientific Truth," in Reinhard Selten, ed., *Rational Interaction: Essays in Honor of John C. Harsanyi* (New York: Springer Verlag, 1992). As a historical aside, John Harsanyi, Reinhard Selten, and John F. Nash, Jr. (of *A Beautiful Mind* fame) shared the 1994 Nobel Prize in Economics for their pioneering work in mathematical game theory. My essay attempted to characterize the meaning of "truth" in six diverse branches of science via a single unifying concept, namely that of the "irreducible symmetry group of a theory," a concept that arises in abstract algebra. Aesthetics was one of these six branches, along with relativity theory, game theory, ethics, information theory, and statistics.

A decade after first developing my theory of aesthetics, I discovered that the great Harvard mathematician George David Birkhoff had developed a somewhat similar theory about complexity in aesthetics in the late 1920s. However, his theory failed to introduce the critical distinction between the relative complexity of theme versus transformation, probably the principal reason why his theory has been deemed unsuccessful. Scholars such as Gary Greenfield have recently suggested that Birkhoff's measure of complexity is in fact a measure of "disorder," not of aesthetic success or beauty.[6]

Finally, I would like to dedicate this essay to two people: the late George Levy, M.B.E., of London, who was tireless in his support and understanding of my addiction to the pursuit of aesthetic excellence; and the late Tracey Albainy of the Museum of Fine Arts, Boston, who encouraged me to write up my theory of aesthetics in simple terms, who was the architect of the exhibition that this book accompanies, and who remains its inspiration.

NOTES

1. More specifically, any sufficiently regular geometric theme can be defined as the zero locus of an ideal of polynomial functions. In simpler cases, this ideal will be principal, and the complexity of the theme will correspond to the order of the generating polynomial. In more general cases, we must use more sophisticated measures such as Shannon or Kolmogorov-Chaitin complexity.

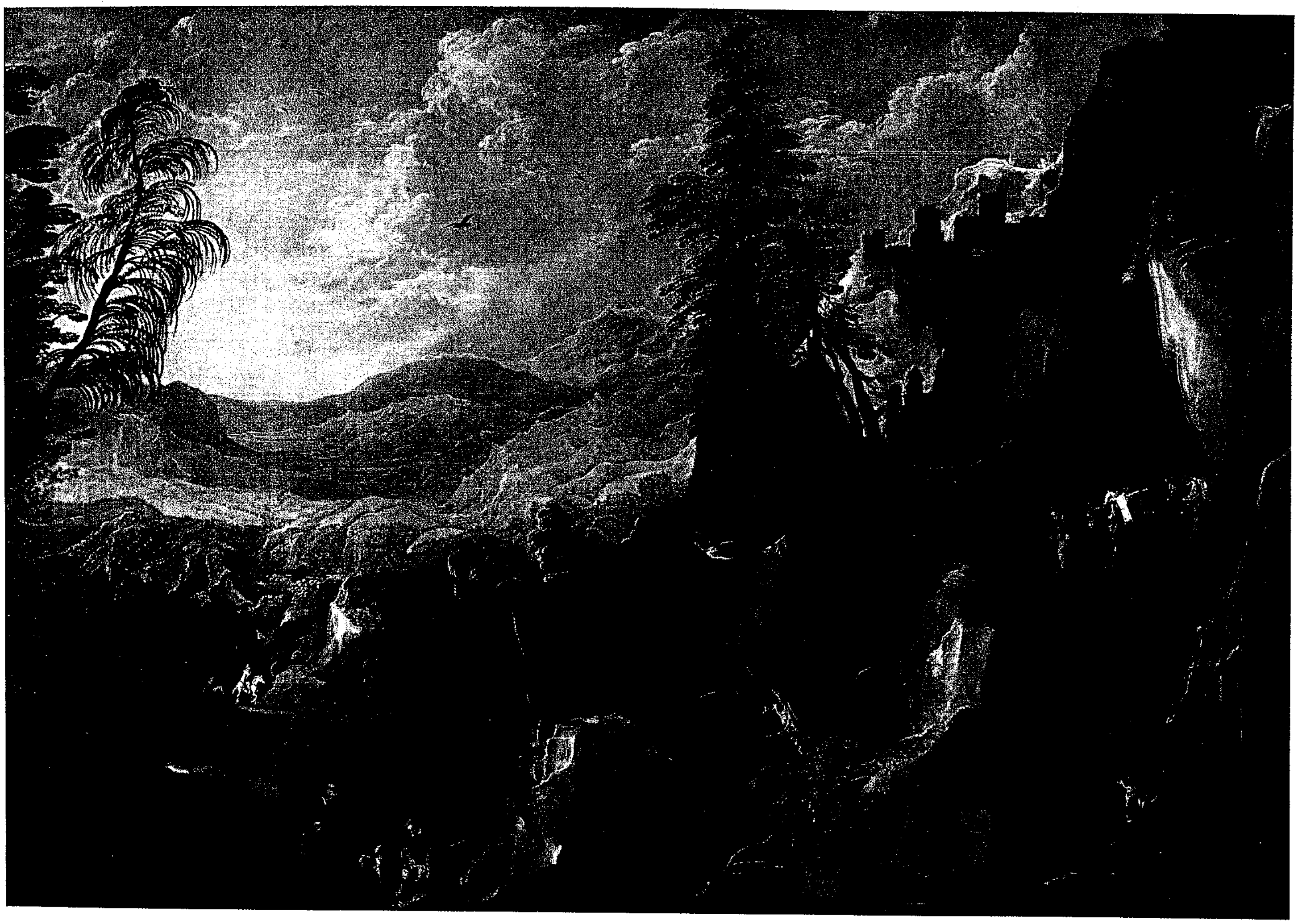
2. There are, in fact, two dimensions of complexity that arise here: first, the number of transformations needed to create the echoes of the designated theme, and second, the degree of complexity of each individual transformation. The overall measure of complexity will thus increase with the number of transformations, and with their average complexity.

3. A superb history of this tradition in mathematical aesthetics is found in Matila Ghyka's *The Geometry of Art and Life* (1946; reprint, New York: Dover Publications, 1977).

4. This form of context-dependence is similar to that lying at the heart of general relativity theory in physics: the apparent degree of complexity of a given design element (theme) in art is *continuously rescaled according to its context*, exactly as the strength of the gravitational field is continuously rescaled by the degree of space-time curvature at each point in the space-time manifold. All this is possible because both the present aesthetic theory and relativity are nonlinear in the fundamental sense that everything within them is *mutually coupled*—that is, everything depends upon everything else. The importance of this concept in science was first promulgated by the physicist Ernst Mach, and it played a very important role in shaping Einstein's thinking.

5. In the case of the Sydney Opera House, a highly complex lobe-shaped "sail" theme is transformed in the simplest possible manner, by elementary transpositions. In the case of the Transamerica Pyramid, the two "ears" atop the building when inverted and joined together—and then magnified—perfectly replicate the entire building without its ears. The remaining examples cited conform equally well to the present theory.

6. Specifically, Greenfield wrote: "In retrospect, Birkhoff's introduction of the aesthetic metric $M = O/C$, where O is order and C is complexity, and its subsequent application to evaluating pleasing polygons and elegant vases, seems to be more about measuring orderliness than about assigning any aesthetic measure to creative works that would be of artistic interest, but it does clearly mark the beginnings of computational aesthetics"; see "On the Origins of the Term 'Computational Aesthetics,'" in Laszlo Neumann, Mateu Sbert, Bruce Gooch, and Werner Purgathofer, eds., *Computational Aesthetics 2005: Eurographics Workshop on Computational Aesthetics in Graphics, Visualization, and Imaging* (Wellesley, MA: A. K. Peters, 2005), 9.



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