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Title: Using Magic in the Teaching of Probability and Statistics

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Using magic in the teaching of statistics

Abstract. This paper explores the role magic tricks can play in the teaching of probability and statistics, especially for lectures in college courses. Demonstrations are described that hit many families of topics, including: basic probability and combinatorics, distributions, hypothesis testing, and advanced topics such as Markov chains. Possible benefits identified include student engagement, a focus on conceptual understanding, development of critical thinking, and an opportunity to reflect upon the role of assumptions and estimates of probabilities.

Keywords: hypothesis testing, magic, pedagogy, *p*-value, probability, statistics

"A mathematician is a conjurer who gives away his secrets."

– John Conway, quoted in Havil (2008, p.131)

1. Introduction

Magic is one of the 20 modalities of fun identified by Lesser & Pearl (2008) as having potential for motivating students in statistics courses. The use of magic has recently been gaining attention in advancing areas of science such as cognitive neuroscience (Martinez-Conde & Macknik, 2008), biology (Kuhn & Land, 2006) and cognitive psychology (Kuhn, Amlani & Rensink, 2008). Several papers and books are available on classroom uses of magic involving mathematics, especially elementary algebra (e.g., Edwards 1992, 1994; Matthews, 2008). However, there appear to be no books and only a few isolated articles (e.g., Mansfield, 1989) presenting the use of magic to explain concepts in probability or statistics.

The lack of instructional guides for incorporating magic into probability and statistics lectures is surprising because magic tricks are designed to be low-probability events in the eyes of a spectator, so that a probability/statistics instructor can take advantage of producing (what appear to be) low-probability events to enhance the intuition and understanding of probability. Magic catches our attention, and therefore has the potential to engage the mind of a probability/statistics student precisely because (it appears) an unlikely event has happened that seems difficult to explain by chance alone. Predicting the result of a coin toss ($p = 0.50$) would not be impressive enough to capture the attention of a student, but predicting a card drawn from a 52-card deck would ($p < 0.02$). Furthermore, probability and statistics have more than their share (relative to mathematics, for example) of results that are surprising or counterintuitive, and thus offer a natural vehicle for connections to magic.

Incorporating magic tricks into the probability and statistics classroom can enhance instruction in several ways. First, magic tricks are one way to include visual demonstrations, and provide an opportunity for the instructor to rely less on textbook and blackboard-oriented teaching techniques (Felder, 1993). Second, magic tricks in the classroom are often class-participatory demonstrations, which can be effective tools in communicating probability and statistics concepts (Gelman & Nolan, 2002). Such demonstrations are likely to involve active learning and emphasis on conceptual understanding, characteristics which are called for by the American Statistical Association (2005). Third, as stated above, all magic effects are based on a simple premise: Given a spectator's assumptions, an unlikely event occurs by the conclusion of the trick, and the surprise a student experiences can help to aid his/her understanding of probability. In a large set of magic tricks (e.g., involving making selections or predictions from a set of cards or other objects), the probability of the unlikely event can be quantified, and these are the prime candidates for inclusion as probability and statistics demonstrations.

This paper presents several example magic tricks we have used in our classes to illustrate concepts and calculations in probability and statistics lectures. While several are examples already described in the literature, we include additional discussion and novel applications. All of the tricks involving props can be purchased at physical or online magic shops. We group the demonstrations by the topic whose concepts they help illuminate: basic probability and combinatorics, distributions, hypothesis testing, and advanced topics.

2. Demonstrations

2.1 Basic probability and combinatorics demonstrations

Early on in our introductory probability and statistics courses, we explain how to compute the probability of the intersection of independent events. We have found that Hen Fetsch's 1954 magic trick "Mental Epic" to be an effective supplement to more conventional illustrations of this topic. The trick involves a portable chalkboard that is divided into six separate regions in two rows and three columns and with "covers" to seal each of the three predictions. There are videos online that demonstrate the prop, such as http://www.youtube.com/watch?v=4wxlKl5lCiw. (As an aside, we have also used a low-tech version of this effect described by Einhorn (2002) that does not depend on using a particular commercially-available prop.) The instructor brings a 52-card deck, a 6-sided die, and a coin. The instructor explains that he will attempt to predict the outcomes of a coin flip, a die roll, and a card selection. He explains that he will first predict the result of a coin flip. In the upper left square on the board, the instructor writes down his prediction and covers it with an opaque piece of cardboard. A randomly selected student is then asked to flip a fair coin, and write down the result of the coin flip in the region on the board below the covered prediction. Without showing the first prediction, the instructor then explains he will write down his prediction of a die roll, which he does in the upper middle region of the board, and then covers it. A randomly selected student then rolls a fair die and writes down the face in the lower middle region of the board. Finally, the instructor says he will predict a card chosen from a deck. The instructor writes the prediction in the upper right corner and covers it with a piece of cardboard. A student then selects a card from the deck, and writes down the selection below the covered prediction. At this point, we find it appropriate to discuss with our students the individual probabilities of correctly predicting the coin flip, die roll, and card selection, followed by formally revisiting the calculation of the probability of the intersection of all three: $1/2 \times 1/6 \times 1/52 = 1/624$. We then reveal that, in fact, all three predictions are correct, so that their sense of surprise is consistent with the small probability of the event that occurred. This effect can serve as a starting point for related computations, including the computation for the probability of at least one of the three predictions being correct, the probability of certain combinations of predictions being correct, and so on. We note that the original "Mental Epic" trick can be used to make three predictions of any type, and that by choosing the predictions to involve coins, dice, and cards, we have quantifiable probabilities that are suitable for the context of a statistics class.

To illustrate the probability of the intersection of events that are not independent using the multiplication rule with telescoping conditional probabilities, we have found an effective demonstration to be "Mental Image," the 1958 trick by Dr. (Stanley) Jaks. Similar tricks can be substituted that involve making several simultaneous predictions. This trick can also be used to demonstrate counting permutations. The trick involves five large distinct ESP cards – one showing wavy lines, one with a plus sign, one with a square, one with a star, and one with a circle. A randomly selected student is asked to seal each card in separate opaque envelopes, shuffle the envelopes, and hand the stack of five envelopes to the instructor. The instructor explains that he will predict the card in each envelope before opening any of them. To compute the probability of this prediction, we let A_1 be the event that the first prediction is correct, A_2 be the event that the second prediction is correct, and so on. We then discuss how to find the probability of the intersection of all five events if the predictions were made randomly, and have the students recall or verify that the probability is easily computed as

$$P(A_1)P(A_2 \mid A_1)P(A_3 \mid A_1, A_2)P(A_4 \mid A_1, A_2, A_3)P(A_5 \mid A_1, A_2, A_3, A_4),$$

which is $(1/5)(1/4)(1/3)(1/2)(1) = 1/120$.

We then make our predictions, usually writing them on the outside of the envelopes. In sequence, we reveal that each prediction is indeed correct. After each prediction is correctly made, we point to each term in the telescoping conditional probabilities to clarify how we obtained the values. For example, after correctly predicting the first card, we explain that $P(A_2 \mid A_1)$ is ¼ if we are simply guessing because after correctly guessing the first card, four cards remain so that the probability of a correct guess is 1 out of a reduced sample space of four. Note that this scenario requires students to have a conceptual interpretation of conditional probability, rather than performing formal calculations for each component factor, in support of the third recommendation of ASA (2005).

It is worth noting that this demonstration can also be used to motivate the calculation of permutations. After the instructor guesses the card in each envelope, a discussion can take place where it is of interest to know the number of possible arrangements of five cards among the five envelopes. If the instructor is merely guessing, then the specific guesses he made is one particular arrangement out of a total of 5!=120 possible. This is an alternative way to arrive at the event probability of 1/120.

The “Birthday Problem” (see Lesser, 1999 for a discussion) can serve as a good example of the misperception of probability disguised as a magic trick. The calculations are straightforward but interesting enough for our more advanced introductory classes to warrant this demonstration. Assuming we have a sufficiently large class (e.g., at least 35 students), we tell students we are going to perform a magic trick, and ask students to create a mental image of their birthday. After appropriate theatrics, we proclaim that two of the students have the same birthday. The common initial belief among students is that it would take around 183 people to have even a 50% chance (Lesser, 1999), when the actual number necessary is only 23. We ask students to state their birthday out loud as we go around the room, and stop once we obtain two students with the same birthday (which is highly likely to happen with our class size). Once a match is found, we explain that the probability of finding a match is much more likely than it would seem at first, and proceed to demonstrate the computation. This involves finding the probability of the complement (no two students have the same birthday), which is the telescoping product of conditional probabilities as in the ESP card prediction example.

Two points about the Birthday Problem “trick” are worth mentioning. First, unlike most of the tricks in this paper (where the event that appeared to happen has a very low probability), this trick involves students being surprised by the occurrence of an event that only appears to be unlikely but actually is not. Research (e.g., Shaughnessy 1977, 1992) shows that people generally tend to underestimate disjunctive probabilities (i.e., probabilities that something happens “at least once”). Second, it is possible, though unlikely, that this trick will not “work” and teachers need to be prepared to turn that outcome into a mini-lesson with more detail about how the probabilities depend on the size of the class and how one might conduct simulations. We note that the birthday problem mathematics has been used to support other magic tricks (e.g., Mulcahy, 2006).

2.2 Demonstrations involving distributions

We have employed demonstrations to illustrate the concept of distributions of functions, and of sampling distributions. For example, the “Divining Rod” trick in Edwards (1994, pp. 28-31) can be described as the student finding all possible permutations of her secretly chosen three-digit number with distinct digits, and then dividing the mean of those (six) permutations by the sum of the digits of the original three-digit number. We explain that depending on the choice of the 3-digit number, a different computation might result. If the student selected her original three-digit number at random, then the distribution of the resulting computation is the distribution of that function relative to a discrete uniform distribution on three-digit numbers with non-repeating digits. What is not apparent to the student is that the distribution is degenerate – it is a point mass at 37. We then find a creative way to reveal that the computation results in the value 37 (which students can later use algebra to verify that this is always the answer).

A slightly more elaborate version of the preceding demonstration involves the magic trick "Predict Perfect" by Meir Yedid. The instructor shows nine cards having the digits 1 through 9 and also shows an envelope containing a prediction which the instructor places in full view during the demonstration. After shuffling these cards, the instructor gives three students three cards each and tells the students to shuffle their three cards. The instructor explains that he will form three 3-digit numbers based on the values on the cards, and then add the three numbers. He asks the three students to call out the first of the three shuffled set of cards they are holding – these will be the hundreds digits of the three 3-digit numbers. Then the instructor asks the students to call out the values of the second cards, which will be the tens digits. Finally, the instructor asks for the values of the third cards, which become the ones digits of the 3-digit numbers. The instructor then sums the three 3-digit numbers. Before revealing the prediction, the instructor demonstrates that different configurations of the nine cards would produce different final sums. Depending on the class, we go through the details of understanding the entire sampling distribution of the sum, showing for example that there are a total of 199 unique sums possible (noting that the smallest possible sum is 774, the largest is 2556, and all sums must be divisible by 9). We also explain how to compute the probability of certain sums based on the process being entirely random, and show which sums have the greatest probability (specifically, 1566, 1575, 1638, 1656, 1674, 1692, 1755, and 1764, each of which can be shown with effort to have a 3/280, or 1.07%, probability – the calculations to determine this probability is a brute-force computation, and requires enumerating all outcomes of the sample space within a computational software package). We then finally reveal the prediction, which of course is correct, and illustrate that even though it was unlikely to guess the correct sum, there were multiple outcomes in the original sample space that would have led to the predicted sum. For example, if the predicted sum were 1647, then we explain that several outcomes could have led to this sum, for example 124+536+987, or 371+582+694, or 213+645+789, among others. We therefore illustrate that even if the 9 digits were arranged randomly to construct three sets of 3-digit numbers, the distribution of the sum of the 3-digit numbers is not uniformly distributed. We find this concept helpful to establish through such a demonstration before teaching topics such as the sampling distribution of the sample mean applied to, for example, rolls of two dice.

2.3 Hypothesis testing demonstrations

Castro Sotos et al. (2009) review many misconceptions students are known to have about the meaning of a *p*-value, such as mistakenly believing (despite the efforts of the instructor) that it is the probability of the null hypothesis. Because the *p*-value is a probability of observing data at least as extreme as what was observed under the null hypothesis, using magic tricks to produce low-probability events is an ideal method for illustrating the concept of a *p*-value. Lesser (2006) has noted that by using a procedure or prop from magic to generate an unlikely event, student probability intuition can be discussed and connected to the formal idea of a *p*-value.

Lesser (2006) discussed the “Predicting Coin Flipping” example of Maxwell (1994), in which students have the chance to notice at what point they became suspicious when a flipped coin is correctly predicted every time. Maxwell described the instructor as doing the flip while a student predicts, and then the instructor always reports (lying as necessary) that the prediction was correct. Holland (2007) offers a variation involving students discussing at what point they became suspicious that something was “fake” or “rigged” when a flipped coin kept landing on heads (the instructor was indeed using a two-headed coin, a prop commonly available in magic stores; because students may have heard of this prop, it may be even more effective to buy a “two-tailed” coin to use). This discussion gives context for significance level, as each student personally experienced their threshold of suspicion.

Whether the setup involves tracking which outcomes were heads or which outcomes involved correct predictions, the context can be conventionally framed as testing the null hypothesis that a binomial probability equals ½. We find that most students begin to get suspicious after the fifth head or tail in a row. By showing students that the probability that the first five tosses are all heads or all tails works out to be 1/16 = 0.0625, which is close to the conventional benchmark of 0.05, which supports statistician Fred Mosteller’s claim (Cohn, 1989, p. 20) of “some empirical evidence that the rarity of events in the neighborhood of 0.05 begins to set people’s teeth on edge.” This demonstration also is a vehicle to discuss what it means for a process to be independent or memoryless.

Gelman & Glickman (2000) introduce a fairly simple card trick called the "Invisible Deck." The instructor explains that he has reversed a card in the deck before coming to class. A student is selected at random (for example by having a student catch a piece of chalk we throw in the air), and is asked to name a playing card of his/her choosing. The deck of cards is then fanned with all the cards face-up, and a single card is seen among them face-down. The reversed card is then placed aside in full view of the students, still without revealing its identity. Suppose the student named the eight of spades. The instructor then asks the students if they would be surprised if the reversed card was a black card, like the eight of spades. After the students say they would not be surprised, we add a formalism to the discussion by explaining that under the null hypothesis that the naming of a card was unrelated to the card's reversal, the probability would be ½ that the card would be black. The instructor also asks if the students would be surprised if the card were a spade, if the card were an "eight," and if the card were actually the "eight of spades" and the students verify the respective probabilities of ¼, 1/13, and 1/52. After the students say they would be particularly surprised at the reversed card being the eight of spades, we clarify that if it is the eight of spades then there are two possible conclusions: Either (1) the original assumption that the reversed card and the card named were unrelated is true so that we are observing a low-probability event (p = 1/52) under this assumption, or (2) the original assumption is not correct (i.e., that the card being reversed has some connection to the card being named, corresponding to p > 1/52). The instructor then reveals to the surprise of the students that the reversed card is indeed the eight of spades. We have found this an excellent launching point for a more formal setup of hypothesis testing, and often refer back to this demonstration when outlining the conventional hypothesis testing procedure as taught in conventional steps.

2.4 Advanced topics

While many magic tricks lend themselves to illustrate basic concepts in probability and statistics, greater ingenuity is required to find application to more advanced concepts and methods. We present here two magic effects that apply to topics more likely to have a home in advanced courses.

The first example is fairly well-known. The instructor has a student shuffle a deck of cards, and hand the deck back to the instructor. The student is then asked to think of an integer between 1 and 10. Let a_1 denote this number. The instructor, who holds the deck face down, explains that he will slowly count through the cards, turning each one face up in succession, until he has gone through the entire deck. The student is told to count to the a_1-th card, and note the face value of this "key" card (call this value a_2). We will follow the convention that if the key card is a jack, queen, or king, act as if $a_2 = 1$, 2 or 3, respectively. Then the student is to count a_2 more cards, and note the face value of the key card in this position. The student is to continue the recursion of counting cards indicated by the face value of the card at the position on which a sequence ends until the instructor goes through all of the cards in the deck. The student is told to remember the last key card before the deck has been completely counted through. After a suspenseful moment, the instructor then reveals the identity of the last key card, which presumably on the student could possibly know, and of course is correct. Readers can get a sense of the effect by using the online applet at: http://oldweb.cecm.sfu.ca/cgi-bin/organics/carddemo.pl?mbutton=Shuffle

This "Key Card Trick," which uses the principle of counting due to physicist Martin Kruskal (Haga & Robins, 1997), is a relatively straightforward application of discrete-time Markov chains. While the student selects a number a_1 at the start of the trick, the instructor also does so silently. And while the student follows the recursion explained above, the instructor also follows the recursion while counting through the cards. With high probability, the student's and instructor's sequence of key cards will eventually dovetail, and will therefore end the sequence on the same key card regardless of the first key card in the sequence. (The last key card is the key card that is not followed by enough cards to continue the count.) More specifically, the trick can be formulated as an inhomogeneous Markov chain: If d_i is the distance from the instructor's *i*-th key card to the nearest key card of the student, then $0 \le d_i \le 9$, forming 10 discrete states, where $d_i = 0$ is an absorbing state of the chain. With an unlimited chain, it can be shown that the absorbing state is reached with probability 1, though for finite chains there is a non-zero probability that the demonstration will fail and details on calculating this probability appear in Havil (2008). There are versions of this trick involving a deck of cards (Peterson, 2001; Mulcahy, 2000) or a piece of text (Gardner, 1998; Havil, 2008).

We have developed an application of a magic effect to updating probabilistic knowledge via Bayes' Theorem. The following demo can also be used to provide intuition for the concept of the "odds" of an event, as well as a tool to understand the concept of a Bayes factor. The demonstration is based on the card trick "All Alike" created by José de la Torre in the 1970s and demonstrated in an online video at http://math.bu.edu/people/mg/video/all_alike_glicko.wmv. The trick was subsequently marketed as "Incredible!" by Nick Trost, and can be found described in Trost (2008). For this demo, the instructor shows the class a deck of cards where the backs of half of the cards are blue, and half are red. He spreads out the blue half of the deck face down on a flat surface, and asks a student to push any card forward and then place it on the red half. The instructor then selects a red-backed card from his half, and places on the blue half, which the student now holds. Both student and instructor cut their halves of the deck, and spread out the face down cards, showing that one red card is among the blue backs, and one blue card is among the red backs. The instructor then asks the student (on the count of three) to turn over the odd-backed cards in each half. When he does so, the cards are identical (e.g., they are both the 5 of clubs).

Without proceeding further, the student usually speculates that all of the cards that are face down in both halves are identical (which would naturally explain why the odd-backed cards match). Rather than turn over the face down cards to confirm or deny the student's suspicion, we first formalize the problem. Letting S be the event that the two odd-backed cards match, we consider two possible assumptions about the identities of the face down cards: Let T denote the event that all the cards are identical (the 5 of clubs), and let U denote the event that we started with 26 distinct red-backed cards, and a matching set of 26 distinct blue-backed cards. We will assume that T and U are mutually exhaustive, that is, there is no other possible truth. We explain to the students that we would like to say something about $P(U \mid S)$, the probability that we started with 26 distinct cards in each half given that we observed a match between the odd-backed cards.

We explain that because we are making a probabilistic inference about an assumption, Bayes' rule applies. Specifically,

$$P(U \mid S) = \frac{P(S \mid U)P(U)}{P(S \mid U)P(U) + P(S \mid T)P(T)},$$

where $P(S \mid T) = 1$ and $P(S \mid U) = 1/26$. We also have that $P(U) = 1 - P(T)$, even though we do not know each individually. We explain that it is more convenient to think about the problem in terms of the odds of *U* relative to *T* (possibly given *S*). Defining $\text{odds}(U) = P(U)/P(T)$ and $\text{odds}(U \mid S) = P(U \mid S)/P(T \mid S)$, it is straightforward to confirm that

$$\text{odds}(U \mid S) = \left(\frac{P(S \mid U)}{P(S \mid T)} \right) \text{odds}(U),$$

where the parenthetical term is the Bayes factor, and can be understood as the weight of evidence of *U* relative to *T*. For the card demonstration, we conclude that

$$\text{odds}(U \mid S) = \left(\frac{1/26}{1} \right) \text{odds}(U) = \frac{1}{26} \text{odds}(U).$$

Thus, if we start out believing that it is just as likely that the cards are identical (5 of clubs) as they are different, so that $\text{odds}(U) = 1$, then after seeing the matched cards we are now 26 times more inclined to believe that the cards are all the same.

We finally reveal the face down cards, showing that all the cards are indeed the same, but that they are not the same as the odd-backed cards. In other words, all the face down cards are the 7 of diamonds, while the two selected cards are the 5 of clubs. The students are clearly surprised, and by performing a similar calculation to the above process we show that the Bayes factor for the comparison of all the cards being identical (event *T*) versus having all the face down cards be identical but different to the selected cards is 1/676. Thus, relative to the original explanation the student speculated, the actual explanation is even more surprising (in the sense of a Bayes factor) than if the cards were all distinct within halves.

2.5 Other connections to statistics content

Having already explored several tricks in a fair amount of depth, we now note briefly a few additional tricks or connections that interested reader may wish to explore or develop further. A probabilistic trick involving the entire class besides the Birthday Problem trick could involve some sort of sampling of data in class where the instructor correctly predicts that the most common first (i.e., leftmost) digit of numbers in the dataset will be 1, based on a result for first-digits known as Benford's Law (e.g., Fewster, 2009). For example, Gelman & Nolan (2002) describe a classroom demonstration applied to street addresses sampled randomly from the telephone book. Each student could even bring in completely different datasets and have them all aggregated together to result in Benford's law being demonstrated, where the probability that "*d*" is the first-digit is $\log_{10}(1 + 1/d)$, which is clearly a decreasing function of *d*.

Mulcahy (2007) describes a card trick where the volunteer is "forced" to select a set of four numbers from the set {1, 2, 3, 4, 5, 6, 7, 8} such that her set has the same mean, and even more amazingly, the same sum of squares as well as the set of four numbers that were not selected. Because variance is the mean square minus the squared mean, this means that the datasets also have the same standard deviation. The details of how the trick is carried out is in Mulcahy (2007) and will not be repeated here.

Other connections include using common magician “street bets” as vehicles to train students to be more explicit about specifying sample spaces or conditioning events. For example, the “Three-Card Swindle” in Gardner (1982) has been developed into a classroom demonstration by Gelman and Nolan (2002), while the “Three Shell Game” connects to the Monty Hall problem, about which there has been much literature (e.g., Barbeau, 1993).

3. Discussion

The tricks introduced in this paper span a range of concepts and techniques applicable to classes of varied levels of technical sophistication. Just as Berk (2003) identifies a continuum of low-risk to high-risk uses of humor, there is certainly a similar continuum of magic tricks for classroom use. Some tricks are simple and self-working, while others involve multiple steps, additional practice beforehand, or more concentration (in terms of story-telling, calculation, or even sleight of hand) during performance. We highlight some of the main features of the tricks in this paper, including approximate time to carry out the demonstration, Table 1.

TABLE 1: Overview of Magic Tricks for Teaching Statistics

Trick	Statistical Topic(s)	Difficulty to Learn and Perform	Materials and Class (Time Needed)
Mental Epic	Probability of the intersection of independent events	Medium	Mental Epic board, coin, dice, cards. (20-30 minutes)
Mental Image	Probability of the intersection of non-independent events	Medium	Special ESP cards (15-20 minutes)
Birthday Problem*	Probability of the intersection of non-independent events	Easy	Blackboard (15-20 minutes)
Divining Rod	Sampling distribution	Easy	Blackboard (15-20 minutes)
Predict Perfect	Sampling distribution	Easy	Cards with digits 1 through 9 (20-30 minutes)
Predicting Coin Flipping	Hypothesis testing, Binomial tail probability	Easy	"Fake" coin (15-20 minutes)
Invisible Deck	Hypothesis testing	Medium	Special deck of cards (15-20 minutes)
Key Card Trick*	Markov chains	Medium	Regular deck of cards (20-30 minutes)
All Alike	Bayes' Theorem, odds, Bayes factor	Medium	Special deck of cards (20-30 minutes)

In our classrooms (and this paper), we generally choose not to reveal the full secrets of magic tricks out of professional courtesy to working magicians, and because the secrets to the tricks are usually not relevant for (and could distract attention from) an enlightening discussion of probability and statistics concepts and techniques. (Of course, sufficiently interested students know they can always learn magic trick secrets by going to magic books or magic shops.) The intentionality of not revealing a trick can itself become a teaching moment to explain about the reality of inferential statistics. As Holland (2007) notes in further discussion about the (two-headed) coin flipping demonstration, “The coin is safely in your pocket, and the students will never see it. This is just like testing of hypotheses in research situations: you never know the ultimate truth, you just draw conclusions based on the probabilities of your observations.”

Paralleling Gelman & Glickman (2000), our magic trick demonstrations are designed to involve students in traditional lecture material, but are not meant to be a replacement for student-initiated investigations. In fact, students may enjoy the challenge (for enrichment or extra-credit) to devise their own magic tricks to illustrate particular topics. Our demonstrations also should be a vehicle to support critical thinking throughout the course. For example, in magic tricks and in real-life data, what initially appears to be a random sample (or random selection) is actually not, and it is no small feat to develop the skills and habit of mind to engage in this mode of critical thinking.

While it is hoped that we have made the case for how magic tricks can facilitate a memorable experience of conceptual aspects of the discipline, a concern can be raised that students may have a negative feeling of having been “tricked.” To the extent magic tricks produce surprising results, they are similar to paradoxes or counterintuitive examples in statistics (e.g., Lesser 1999, 2001). Educators need to be aware of the possible pitfall expressed by former National Council of Teachers of Mathematics president Gail Burrill (1990), repeated by the ASA (1994) that emphasis should be on building intuition, not on "probability paradoxes or using statistics to deceive." Falk and Konold (1992) express a similar caution that students may despair if teachers persist in exposing their vulnerabilities.

However, the stronger trend in research studies on this issue is that well-chosen paradoxes are more likely to motivate than demoralize (e.g., Lesser 1998; Shaughnessy, 1977; Wilensky, 1995). In short, magic tricks must be thoughtfully chosen so that the quality and intensity of surprise is "just right," so that students will feel engaged to want to dig deeper to discuss and understand, rather than feel unduly tricked or deceived. Indeed, some introductory textbooks (e.g., Utts, 2005) actually address explicitly how intuition can be fallible or distorted by such phenomena as the availability heuristic, anchoring, the representativeness heuristic, the conjunction fallacy, and forgotten base rates. And the classic by Huff (1993) on statistical literacy and critical thinking playfully incorporates the "deception" theme in its title: *How to Lie with Statistics*. As researchers continue to learn about the cognitive and physiological dynamics in the brain in response to seeing magical happenings, educators (particularly in probability and statistics) can continue to take advantage of the unique opportunities afforded in enhancing the concepts of statistics in a way that students experience the discipline as "magical" (in a positive sense) in its power to illuminate the unknown.

References

American Statistical Association (2005). *Guidelines for assessment and instruction in statistics education: College report*. http://www.amstat.org/education/gaise/GAISECollege.htm

American Statistical Association (1994). G. Burrill (Ed.), *Teaching statistics: Guidelines for elementary to high school*. Palo Alto, CA: Dale Seymour.

Barbeau, E. (1993). Fallacies, flaws and flimflam. *College Mathematics Journal, 24*(2), 149-154.

Berk, R. A. (2003). *Professors are from Mars, students are from Snickers: How to write and deliver humor in the classroom and in professional presentations*. Sterling, VA: Stylus Publishing.

Burrill, G. (1990). Implementing the Standards: Statistics and probability. *Mathematics Teacher*, *83*(2), 113-118.

Castro Sotos, A. E., Vanhoof, S., Van den Noortgate, W., & Onghena, P. (2009). How confident are students in their misconceptions about hypothesis tests? *Journal of Statistics Education, 17*(2), http://www.amstat.org/publications/jse/v17n2/castrosotos.html

Cohn, V. (1989) *News and numbers: A guide to reporting statistical claims and controversies in health and other fields*. Ames, IA: Iowa Sate University Press.

Edwards, R. (1992). *Algebra magic tricks: Algecadabra!* (Vol. 1). Pacific Grove, CA: Critical Thinking Press and Software.

Edwards, R. (1994). *Algebra magic tricks: Algecadabra!* (Vol. 2). Pacific Grove, CA: Critical Thinking Press and Software.

Einhorn, N. (2002). *The art of magic and sleight of hand*. London: Lorenz Books.

Falk, R., & Konold, C. (1992). The psychology of learning probability. In F. Gordon & S. Gordon (Eds.), *Statistics for the twenty-first century* (pp. 151-164). Washington, DC: Mathematical Association of America.

Felder, R. (1993). Reaching the Second Tier: Learning and Teaching Styles in College Science Education. *J. College Science Teaching*, 23 (5), 286-290.

Fewster, R. M. (2009). A simple explanation of Benford's law. *The American Statistician*, *63*(1), 26-32.

Gardner, M. (1998). Ten amazing mathematical tricks. *MAA Math Horizons*, *6*(1), 13-15, 26.

Gardner, M. (1982). *Aha! Gotcha: Paradoxes to puzzle and delight*. New York: W. H. Freeman.

Gelman, A., & Nolan, D. (2002). *Teaching statistics: A bag of tricks*. New York: Oxford University Press.

Gelman, A., & Glickman, M. E. (2000). Some class-participation demonstrations for introductory probability and statistics. *Journal of Educational and Behavioral Statistics, 25*(1), 84-100.

Haga, W. & Robins, S. (1997). On Kruskal's principle. In J. M. Borwein, P. Borwein, L. Jörgenson, R. Corless (Eds.), *Organic mathematics*: *Proceedings of the Organic Mathematics Workshop* (pp. 407-412). Providence, RI: American Mathematical Society.

Havil, J. (2008). *Impossible? Surprising solutions to counterintuitive conundrums*. Princeton, NJ: Princeton University Press.

Holland, B. K. (2007). A classroom demonstration of hypothesis testing, *Teaching Statistics, 29*(3), 71-73.

Huff, D. (1993). *How to lie with statistics*. New York: W. W. Norton

Kuhn, G., Amlani, A. A., & Rensink R. A. (2008). Towards a science of magic. *Trends in Cognitive Science. 12* (9), 349-354.

Kuhn, G. & Land M. K. (2006). There's more to magic than meets the eye. *Current Biology*, 16, 950-951.

Lesser, L. (1998). Countering indifference using counterintuitive examples. *Teaching Statistics*, *20*(1), 10-12.

Lesser, L. (1999). Exploring the birthday problem with spreadsheets. *Mathematics Teacher, 92*(5), 407-411.

Lesser, L. (2001). Representations of reversal: Exploring Simpson's paradox. In A. Cuoco & F. Curcio (Eds.), *The roles of representation in school mathematics* (pp. 129-145). Reston, VA: National Council of Teachers of Mathematics.

Lesser, L. (2006). Making statistics learning fun, invited webinar, Consortium for the Advancement of Undergraduate Statistics Education. Archived at: http://www.causeweb.org/webinar/teaching/2006-04/.

Lesser, L. & Pearl, D. (2008). Functional fun in statistics teaching: Resources, research, and recommendations. *Journal of Statistics Education*, *16*(3), 1-11. http://www.amstat.org/publications/jse/v16n3/lesser.html.

Mansfield, E. R. (1989). Teaching elementary probability like magic. *Proceedings of the Statistical Education Section* (pp. 87-88). Washington, DC: American Statistical Association.

Martinez-Conde, S. & Macknik, S. L. (2008). Magic and the brain. *Scientific American, 299*(6), 72-79.

Matthews, M. E. (2008). Selecting and using mathemagic tricks in the classroom. *Mathematics Teacher, 102*(2), 98-101.

Maxwell, N. P. (1994). A coin-flipping exercise to introduce the *p*-value. *Journal of Statistics Education 2*(1), http://www.amstat.org/publications/jse/v2n1/maxwell.html.

Mulcahy, C. (2000). Mathematical card tricks. http://www.ams.org/featurecolumn/archive/mulcahy1.html

Mulcahy, C. (2006). Card Colm: Better poker hands guaranteed. *MAA Online*, http://www.maa.org/columns/colm/cardcolm200606.html.

Mulcahy, C. (2007). Card Colm: Plurality events, standard deviations, and skewed perspectives. *MAA Online*, http://www.maa.org/columns/colm/cardcolm200712.html

Peterson, I. (2001). Guessing cards, *MAA Online*, http://www.maa.org/mathland/mathtrek_12_24_01.html.

Shaughnessy, M. (1977). Misconceptions of probability: An experiment with a small-group, activity-based model building approach to introductory probability at the college level. *Educational Studies in Mathematics* 8: 295-316.

Shaughnessy, M. (1992). Research in probability and statistics: Reflections and directions. In Douglas A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp.465-494). New York: Macmillan.

Trost, N. (2008). *Subtle card creations*, vol.1. Humble, TX: H & R Magic Books.

Utts, J. M. (2005). *Seeing through statistics* (3rd ed.). Belmont, CA: Thomson Brooks/Cole.

Wilensky, U. (1995). Paradox, programming and learning probability: A case study in a connected mathematics framework. *Journal of Mathematical Behavior*, 14, 253-280.