The use of puzzles in teaching mathematics
Author(s): JEAN PARKER
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The use of puzzles in teaching mathematics

JEAN PARKER, Central High School, Florence, Alabama.

Puzzles always intrigue high school classes and, hence, form an excellent means of motivating the study of mathematics. Here is an excellent collection of usable puzzles for the high school classroom.

Since antiquity people of all ages have found pleasure in puzzles, tricks, and curiosities of all kinds. The problem of the fox, the goose, and the peck of corn and how to get them across the river was known in the time of Charlemagne, about 800 A.D.¹ The hare and hound problem appears in an Italian arithmetic of 1460, and many other present-day puzzles have come down to us from a much earlier time.²

Some of the earlier puzzles have lost much of their original signification or interest inasmuch as the subsequent development of newer branches of mathematics afforded simple solutions which deprived them of their initial mystery and appeal. In other cases, the attention given to these perplexing puzzles ultimately led to an extension or clarification of new mathematical fields. One central fact stands out, however: the appeal of puzzles is as irresistible today as it was two hundred or two thousand years ago.

In the voluminous amount of literature which has been produced concerning the use of puzzles in the classroom, there has been attributed to puzzles every conceivable objective, from thought-provoking recreation to a means of improving attitudes, except as a method of teaching mathematics. Many well-known writers have dealt with this field, including, among others, W. W. R. Ball, H. Dudney, H. Schubert, and S. I. Jones. In general, the consensus seems to be that puzzles are excellent devices for securing the attention of a group, as material for clubs and contests, as part of an enriched program for the bright pupil, as a reward after a certain amount of good work, and for pure enjoyment.

However, there are some few who object to the use of puzzle material in the classroom for any objective. For instance, M. C. Bergen contends that there are only two groups in high school mathematics classes today, the interested and the uninterested.³ He says, further, that recreations will provide motivation for neither group, since (1) the interested ones need no motivation, and (2) the uninterested ones will certainly get nothing from a problem purposely worded in a tricky and evasive way. If we accept Bergen’s hypothesis, we may or may not accept his conclusion. But how can we accept a hypothesis that there are only two classes of mathematics students, when we know that there exist varying degrees of interest among pupils. This fact alone tends to invalidate Bergen’s statement.

In opposition to Bergen, there are some who believe that a good teacher tries to secure the attention of his students by presenting the subject they are studying in an

² Ibid.

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attractive form, and to instill its principles, if possible, by processes agreeable to the student. There are, indeed, some teachers who are willing that their students should be not only interested, but entertained, if through such entertainment the real work of the classroom can be advanced.

In support of this theory a well-known study by R. B. Porter has been made to determine the effect of the study of mathematical recreations on achievement. Three experiments were carried out, in which the control-group technique was used. The problems used in the experimental group were carefully selected so that they were related to the work studied at the time. The results of these studies indicate that pupils not only may have fun in mathematical recreations of the puzzle nature without any deterring effect upon achievement, but may achieve more in the process. Increase in achievement, however, was only one of many factors in favor of the experimental group as determined by observation of the classes such as:

1. The time spent in recreational activities does not inhibit the covering of a prescribed course content; rather, it provides an opportunity for covering much additional material.
2. The use of recreational material is stimulating to the teacher as well as to the pupil.
3. Recreational items may be synchronized with daily assignments, thereby providing a common meeting ground for student and teacher from which a learning situation may progress.
4. Creativeness is encouraged in the pupil as he devises his own recreational items.
5. A form of research is encouraged through the use of recreational material; pupils who have not previously used library facilities have been observed to do so in search for such items.

Certainly any activity which results in the aforementioned advantages would be worth while in itself. But the primary result of this experiment is the conclusive indication that real mathematics can be taught through puzzles, wrinkles, and trick problems.

It is my contention that puzzles can be made to serve dual purposes:
1. To secure the interest and attention of the group.
2. To teach mathematics by illustrating and clarifying certain mathematical concepts and techniques, by securing a higher mastery of subject matter, by developing skill in manipulation, by making mathematical learnings more permanent, and by developing an appreciation of the systematic approach of algebraic methods.

It is my purpose to present a sampling of the available puzzle material and methods of using this material in the teaching of mathematics.

An unfailing source of entertainment for the mathematics class is found in the multitude of mathematical problems in which an apparently correct chain of operations leads to an absurd result. These problems also have considerable value in the teaching of mathematics. Not only do they provide a needed relief from day-to-day tasks, but the associations are an aid to memory in recalling important facts which have been temporarily forgotten. No formal statement that division by zero is not permissible will impress the pupil to the same extent that he is shocked when he first sees a "proof" by algebra that $5 = 7$ or $1 = 2$.

Almost everyone who has been exposed to elementary algebra has at one time or other seen a proof that any two unequal numbers are equal.

Paradox 1: To prove that any two unequal numbers are equal:

Suppose that

$$a = b + c$$

$$a = c + b$$

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where $a$, $b$, and $c$ are positive numbers. Then inasmuch as $a$ is equal to $b$ plus some other number, $a$ is greater than $b$. Multiply both sides by $a - b$.

Then

$$a^2 - ab = ab + ac - b^2 - bc.$$  

(2)

Subtract $ac$ from both sides:

$$a^2 - ab - ac = ab - b^2 - bc.$$  

(3)

Factor:

$$a(a - b - c) = b(a - b - c).$$  

(4)

Divide both sides by $a - b - c$. Then

$$a = b.$$  

(5)

Thus $a$, which was originally assumed to be greater than $b$, has been shown to be equal to $b$.

Paradox 2: To prove any number is equal to any other.*

Let $a$ and $b$ be any two given numbers. Let $c$ and $d$ be two equal numbers. Then

$$c = d$$  

(1)

$$ac = ad$$  

(2)

and

$$bc = bd$$  

(3)

$$ac - bc = ad - bd$$  

(4)

$$ac - ad = bc - bd$$  

(5)

$$a(c - d) = b(c - d)$$  

(6)

$$a = b.$$  

(7)

These exercises may be introduced when a pupil attempts to divide a number by zero. Now the pupil may ask, “Why can’t we divide by zero?” The answer involves the notion of consistency. Division in mathematics is defined by means of multiplication. To divide $a$ by $b$ means to find a number $x$ such that $b \cdot x = a$, whence $x = a/b$. Division by zero then leads either to no number or to any number. Thus, if we admit zero as a divisor, any two numbers can be shown to be equal. Whether or not we agree that it is important to know division by zero is not allowed, we must agree that here is an opportunity to teach some of the underlying principles of our number system, and to impress students with the “reasonableness” of our definitions, axioms, and postulates (e.g., if division by zero is defined, inconsistencies would occur elsewhere in our laws of operation).

Division by zero is sometimes fairly well disguised. For example in the theory of proportions, it is easy to prove that if two fractions are equal, and if their numerators are equal, then their denominators are equal. That is, from $a/b = a/c$ it can be inferred that $b = c$. This inference is not valid for $a = 0$, since step 3 involves division by zero if $a = 0$.

Given:

$$a/b = a/c$$  

(1)

$$ac = ab$$  

(2)

$$c = b.$$  

(3)

Consider the following problem in this light. It is desired to solve the equation:* \[
\frac{x+5}{x-7} - 5 = \frac{4x-40}{13-x}. \]  

(1)

Combining terms:

$$\frac{x+5-5(x-7)}{(x-7)} = \frac{4x-40}{13-x}.$$  

(2)

Simplifying:

$$\frac{4x-40}{7-x} = \frac{4x-40}{13-x}.$$  

(3)

Now since the numerators in (3) are equal, so are the denominators. That is, $7 - x = 13 - x$ or $7 = 13$.

Now, “How do we know that $4x-40$, the numerator on both sides of (3), is not equal to zero?” Here it should be pointed out that the axioms cannot be applied blindly to equations without taking into consideration the values of the variables for which the equations are true. Thus equation (1) is not an identity which is true for all values of $x$, but is an equation which is satisfied only for the value $x = 10$.

* Ibid., p. 86.

* Ibid.

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At this point it would be valuable to introduce problems of this type:

Solve for $x$: \[ (x-3)(x+3) = x^2 - 9 \quad (1) \]

\[ (x-2)(x+1) = x^2 - x - 2. \quad (2) \]

The fact that $x$ vanishes creates a puzzle which eventually leads up to the difference between an identity and a conditional equation.

Another misuse of the axioms is apparent in the following paradox.\textsuperscript{11} In attempting to solve the system of two equations in two unknowns:

\[
\begin{align*}
  x + y &= 1 \\
  x + y &= 2
\end{align*}
\]

we are forced to the conclusion that, since 1 and 2 are equal to the same thing, they must be equal to each other, i.e., $1 = 2$.

What is the error? Again, the values of $x$ and $y$ for which both the equations are true must be taken into account and there are no values of $x$ and $y$ for which $x + y = 1$, and, at the same time, $x + y = 2$. This paradox illustrates a set of inconsistent simultaneous equations in which it is not possible to solve for $x$ and $y$.

Many other algebraic paradoxes involve the extraction of square root. Using an argument which required division by zero we have already proved that any two unequal numbers are equal to each other. Here is a different proof of the same proposition.

Let $a$ and $b$ be two unequal numbers, and let $c$ be their arithmetic mean, or average.\textsuperscript{12}

Then

\[ (a+b)/2 = c, \quad \text{or} \quad a + b = 2c. \quad (1) \]

Multiply both sides by $a - b$: \[ a^2 - b^2 = 2ac - 2bc. \quad (2) \]

Add $b^2 - 2ac + c^2$ to both sides:

\[ a^2 - 2ac + c^2 = b^2 - 2bc + c^2 \quad (3) \]

\[ (a-c)^2 = (b-c)^2. \quad (4) \]

Take the square root of both sides. Then

\[ a - c = b - c \quad (5) \]

or

\[ a = b. \quad (6) \]

What is the error? We neglected the fact that a quantity has $n$th roots. In passing from step (4) to step (5) only the positive signs are used. Had we written (5) as $a - c = -(b - c)$, we should have obtained our original expression, $a + b = 2c$. Here it should be pointed out that the extraction of a square root requires the consideration of both signs, and the one which leads to a contradictory result such as ours must be rejected.

Fallacies are useful in illustrating particular facts. They present those facts in such a manner that they are not easily forgotten. But most important is the fact that fallacies point out the logic and consistency in mathematics. They impress the student with the knowledge that when one obtains absurd results even through seemingly correct operations, there is some logical, concrete explanation which is consistent with the rules of operation.

Geometrical fallacies are very frequently seen. They are usually divided into two types.\textsuperscript{13} The first type involves the use of a false theorem; and the second involves the use of an incorrect construction. The flaw in the proof is obvious if the construction is attempted, but if the figure has been prepared in advance, the average geometry class will be greatly puzzled.

The purpose of the following examples is to show the danger of depending too much on the drawing in working out a proof. They illustrate the pitfalls in the faulty construction of figures, i.e., "Relations which appear to the eye to be correct may be erroneous and misleading."\textsuperscript{14}


\textsuperscript{12} Northrop, op. cit., p. 79.

\textsuperscript{13} Northrop, op. cit., p. 81.


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Paradox 3: Every triangle is isosceles.\footnote{Northrop, op. cit., p. 102.}
Given: any triangle, as $ABC$.
To prove: $ABC$ is isosceles.

![Figure 1](image1.png)

**Proof:**
Let $DE$ be the perpendicular bisector of $AB$ and $CE$ be the bisector of angle $C$, meeting $DE$ at $E$.
From $E$ draw $EA$ and $EB$.
Draw $EG$ perpendicular to $AC$ and $EF$ perpendicular to $CB$.
Then
\[ \triangle ADE \cong \triangle BDE. \]
Hence
\[ AE = BE. \]
\[ \triangle CEG \cong \triangle CEF. \]
Hence
\[ EG = EF \quad \text{and} \quad CG = CF. \]
Therefore,
\[ \triangle AEG \cong \triangle BEF. \]
Hence
\[ GA = FB. \]
Since
\[ CG = CF \]
it follows that:
\[ CG + GA = CF + FB \]
or
\[ CA = CB. \]

Therefore $ABC$ is isosceles. Paradoxes of this type show how easily a logical argument can be swayed by what the eye sees in the figure and so emphasizes the importance of drawing a figure correctly, noting with care the relative position of points essential to the proof.

Paradox 4: To prove that more than one perpendicular may be drawn from a point to a line.\footnote{Ibid.}
Given: $AB$ and point $P$ outside $AB$.
Required: To draw two perpendiculars from $P$ to $AB$.

![Figure 2](image2.png)

**Solution:**
Draw $AP$ and $PB$ and construct circles on them as diameters cutting $AB$ at $M$ and $N$ respectively. Then $PM$ and $PN$ will each be perpendicular to $AB$, since they form right angles inscribed in semicircles.

These examples cannot fail to impress pupils with the importance of careful thinking and observing in a logical demonstration. The chief value of such exercises lies in the use that can be made of them to show the need for accurate constructions in geometry. Although important conclusions may be deduced from a careful examination of a figure, such cases show that observation unsupported by some reliable check is worthless as a means of proving the truth of geometric statements.

The second type of geometric fallacy illustrates the tendency to misquote a proposition. The absurd results of such
proofs should point out the need for familiarity with theorems, axioms, and postulates that are to be used in a deductive argument.

Paradox 5: Any point on a line is the midpoint of the line.\textsuperscript{17}

Given: \( P \) any point of \( AB \).
To prove: \( P \) is the midpoint of \( AB \).

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure3.png}
\caption{Figure 3}
\end{figure}

Proof:

On \( AB \) as a base construct an isosceles triangle \( ACB \) and draw \( CP \). Since \( AC = BC \), \( CP = CP \), and \( \angle A = \angle B \), \( \triangle ACP \) and \( \triangle BCP \) are congruent, and \( AP = PB \) (corresponding parts).

Most boys and girls are thoroughly intrigued by number tricks and age tricks. Much intellectual curiosity and alertness can be aroused in the ordinary boy or girl of high school age through the “mind reading” qualities of the teacher. These problems may be employed in the teaching of mathematics as legitimately and perhaps more effectively as much of the material we are now using, since most of the “mysterious” number problems may be easily explained by the first steps in algebra. They provide needed practice in selecting the essential data from the unessential data, in setting up equations, and in mastering rules of operations and techniques in the solution of equations. Some require very careful reading before they can be translated into the language of simple algebra. This quality will tend to develop what we call “critical” or “logical” thinking. The problems listed below are illustrative of this group.

1. Tell a member of the class to select two numbers and to tell you their quotient and their difference and you will tell him the numbers. How is this done?\textsuperscript{18}

The small number is equal to the difference divided by the quotient decreased by one. If \( a/b = q \) and \( a - b = d \), then it is easily shown by algebra that \( b = d/(q - 1) \).

2. Tell a person to think of a number, to square it, to square the next larger number, and then tell you the difference between the squares and you will tell him the number. How is it done?\textsuperscript{19}

Let \( n \) be the number and \( n + 1 \) the next larger; then the squares are \( n^2 \) and \( n^2 + 2n + 1 \). The difference is \( 2n + 1 \). Hence, subtract 1 and divide by two and you have the number selected.

Puzzles of this type emphasize the advantages of the systematic approach to the solution of problems that is found in algebra.

The ability to square numbers mentally is another elementary algebraic puzzle. Its algebraic explanation affords a highly desirable application of the formulas to arithmetical computation, while at the same time the algebraical statement extends the rule to all numbers between 25 and 150.\textsuperscript{20}

Rules of this kind are excellent for rapid calculation.

The rules are all applications of the simple formulas \( (a+b)^2 = a^2 + 2ab + b^2 \) and \( (a-b)^2 = a^2 - 2ab + b^2 \) and the teacher of algebra should emphasize this kind of simple application of algebraic formulas. Thus

\[
(50 - a)^2 = 2500 - 100a + a^2 \\
(50 + a) = 2500 + 100a + a^2 \\
(100 + a) = 10,000 + 200a + a^2
\]

\textsuperscript{17} Smith and Reeve, \textit{op. cit.}, p. 402.

\textsuperscript{18} Ibid., p. 395.

\textsuperscript{19} Ibid., p. 384.

\[ (20 + a) = 400 + 40a + a^2 \]
\[ (10 + a) = 100 + 20a + a^2 \]
\[ = 10(10 + a + a) + a^2. \]

Another type of puzzle is that frequently found in algebra books under the title of “practical problems.” In most puzzles of this kind, the difficulty lies in the phraseology, while the mathematics is simple. They provide drill in setting down facts in a logical sequence and also teach the solution of equations through problems made interesting by their puzzling nature.

For example:

1. A man is twice as old as his wife was when he was as old as she is now. When she is as old as he is now, the sum of their ages will be 100 years. Find their ages now.\(^{21}\)

The problem involves two equations:

\[ m = 2(w - m - w) \]

and

\[ m + m - w + w + m - w = 100 \]
\[ m = 44 \frac{2}{3} \quad w = 33 \frac{1}{3}. \]

2. Mary is 18 years old. Ann is twice as old as Mary was when Ann was as old as Mary is now. How old is Ann?\(^{22}\)

Let us say that \(x\) years ago Ann was as old as Mary is now; that is, she was then 18.

Then

\[ a - x = 18 \]

and also

\[ a = 2(18 - x) \]

hence

\[ a = 18 + x \]

and so

\[ 18 + x = 2(18 - x) \]

solving,

\[ x = 6 \]

hence

\[ a = 2(18 - 6) = 24. \]

Perhaps the puzzle most frequently discussed by the public is the “trick” puzzle. It is said that the following puzzle caused chaos in banking circles:

Assume we make a deposit of $50 in a bank.

<table>
<thead>
<tr>
<th>Withdraw</th>
<th>Leaving</th>
</tr>
</thead>
<tbody>
<tr>
<td>$20.00</td>
<td>$30.00</td>
</tr>
<tr>
<td>$15.00</td>
<td>$15.00</td>
</tr>
<tr>
<td>$9.00</td>
<td>$6.00</td>
</tr>
<tr>
<td>$6.00</td>
<td>$0.00</td>
</tr>
</tbody>
</table>

$50.00
$51.00

We now present our figures to the bank, showing the discrepancy, and demand the extra dollar. Repeat ten thousand times and retire for a while.\(^{23}\)

Since such problems are so common it would be well if the mathematics teacher could find a way in which she might use them to teach mathematics. In general, these discrepancies result from a lack of observation and critical thinking. A puzzle of this type should arouse the innate curiosity of most high school pupils and should develop (1) the processes of thinking and (2) a questioning attitude toward such misleading results. (Why should the second column add up to the same amount as the first?)

There are many mechanical puzzles, such as the “Fifteen Puzzle” and the “Tower of Hanoi” which have a mathematical basis far above the level of thinking in the secondary school. However, there is enough enjoyment in the interesting stories accompanying these puzzles and in the manipulation of them knowing that there is a mathematical explanation, to warrant their use in the classroom for recreational purposes. It is to be hoped that through such recreational devices as these a true interest in mathematics may be furthered and a desire to continue mathematical study may be developed through the realization that mathematics can be fun.


\(^{22}\) Smith and Reeve, op. cit., p. 399.

\(^{23}\) Ibid., p. 399.
The popularity of the daily newspaper crossword puzzle has probably caused the development of the cross-number puzzle as illustrated below.

**AREAS AND PERIMETERS**

![Crossword Puzzle]

**Horizontal**
1. Perimeter of a 9" square.
2. Area of a 1" square.
3. Area of a rug 9' × 12'.
4. Circumference of a circle with a diameter of 7".
5. Area of a rectangle 16" × 3½".
6. How many sides has a square?
7. Area of a triangle with a base of 16" and an altitude of 14".
8. Same definition as (11) vertical.
10. How many 2" squares can be made to fit into a rectangle 4" by 10"?

**Vertical**
1. Area of a room 30' × 10½'.
2. Area of a field 60 rd. × 10.1 rd.
3. Area of a unit square.
4. Area of a square 15 cm. on a side.

11. Perimeter of a triangle which is 7" on each side.
12. Perimeter of a rectangle 4.8 cm. × 2.7 cm.
13. Perimeter of a lawn which measures 23½ ft. in length and 21¼ ft. in width.

The purpose of the cross-number puzzle is its use as a motivating device which will enable the teacher to re-create interest in solving mathematical problems. It involves the construction of a puzzle which the students can complete by writing in the correct answers to a set of problem exercises. Its mathematical value lies in the interesting manner in which a review of a unit of work may be undertaken and in its use in the regular classroom instead of traditional tests and written exercises.

It appears that we may safely conclude that most of us have a "puzzle instinct" and that the puzzle question at the right time and place will not only make a class more interesting, but can further the primary responsibility of the teacher, i.e., teaching mathematics. Specific instances have been pointed out in which it is possible both to motivate the work and at the same time insure its mastery. Puzzles and paradoxes lend themselves to the teaching of (1) specific facts, e.g., division by zero is not permitted; (2) techniques in problem solving, e.g., selecting pertinent data, establishing equalities, and operations with equalities; (3) basic mathematical concepts, e.g., accuracy in drawings, and familiarity with well-known propositions in geometry; (4) critical thinking; (5) an appreciation of the power of mathematics.

Puzzle material, properly related to class work can, therefore, be made a valuable aid to teaching.

Now, should the teacher of mathematics desire to locate additional material concerning puzzles, the following sources are suggested:

1. Several newspapers and magazines have daily and weekly features of mathe-
metrical puzzles (e.g., *Popular Science* puzzle page).


3. The bibliography listed in this article.

4. The pupils themselves. With guidance and direction, pupils will submit many interesting items.

5. The teacher. With an underlying source of error of a mathematical principle as a basis, endless problems can be created by the teacher who is interested in this type of work.


**BIBLIOGRAPHY**

**Articles on the use of puzzles in the classroom**


**Sources of puzzles**

**Periodicals**


Are postulates in a mathematical system "true"?

If you want to use a mathematical system in a very practical way, such as in surveying and astronomy, must its principles agree with the reality that you observe and measure? The following excerpts will appear startling to those who answer "yes" to these questions:

"To simplify the computations necessary for the determinations of the direction of the meridian, of latitude, and of longitude or time, certain concepts of the heavens have been generally adopted. They are the following:

a) The earth is stationary.
b) The heavenly bodies have been projected outward, along lines which extend from the center of the earth, to a sphere of infinite radius called the celestial sphere.

The celestial sphere has the following characteristics:

a) Its center is at the center of the earth.
b) Its equator is on the projection of the earth's equator.
c) With respect to the earth, the celestial sphere rotates from east to west about a line which coincides with the earth's axis. Accordingly, the poles of the celestial sphere are at the prolongations of the earth's poles.
d) The speed of rotation of the celestial sphere is 360° 59.15' per 24 hours.1


The use of puzzles in teaching mathematics