By Maksim Maydanskiy - based on a session at the San Jose Math Circle.

The Baby Hummer Card trick.

The following card trick appears in the "Mathematical Magics" by Persi Diaconis and Ron Graham. Take some cards and try it! Then try it on your friends. Diaconis and Graham suggets that you can even do it over the phone! (or chat and sms for that matter). Of course you don't *need* to try it to follow along, but I think it would be more fun if you do.

1 You have 4 cards. Look at the bottom card. *Remember it*.

2 Place the top card on the bottom.

3 Turn the current top card face up.

4 Cut the deck by removing some number of cards from the top and putting them on the bottom. The number of cards you take does not matter, but please do not shuffle.

5 Take top two cards, and turn them over together as a unit (and place back on top).

6 Cut the deck again.

7 Turn the top two cards over, as before.

8 Cut the deck one last time.

9 Turn over top two again.

10 If you wish, give it another cut. Now it must really be mixed! No one can find your original card (Do you remember what it was?)

11 Turn over the top card.

12 Put the top card on the bottom.

13 Put the current top card on the bottom.

14 Turn over the top card.

Name your card. Spread out your deck - 3 cards are facing one way, and one "oddball" is facing another way. That's your card!

I think that's pretty amazing! Why does it work? Let's find out.

A simpler trick.

The Baby Hummer is based on another trick, which we will call "Red and black, up and down". It will be easier to understand "Red and black, up and down" first, and come back to the Baby Hummer afterwards.

Red and black, up and down.

1 Start with any even number of cards, half **black** and half red, in an alternating pattern: R, **B** R, **B**, ..., R, **B**.

2 Cut the deck by removing some number of cards from the top and putting them on the bottom. The number of cards you take does not matter, but please do not shuffle.

3 Take top two cards, and turn them over together as a unit (and place back on top).

4 Repeat any of the above two steps - cutting and flipping two - any number of times in any order.

5 Deal out your cards in a row on a table.

6 Turn over every other card (that is, turn over the second, fourth, sixth etc. cards).

Now all red cards are facing one way, and all **black** cards are facing another way!

How does "Red and black, up and down" work?

The analysis.

Let's think what the steps in 'Red and black, up and down" do. There are two parts to the trick - first some number of cuts and pair flips, and then the dealing and flipping. The last part just reverses the orientation of the even cards; so if we can understand the first part, we should be in good shape.

There are two basic moves - the cut and the "flip two" move.

(i) The cut.

The cutting one card just adds 1 to the position of every card (note that the last card becomes the first; if we had $2k$ cards to begin with, mathematicians would say that we take the positions "mod $2k$").

Cutting more than one card, say l cards, will add l to the postition of every card (mod $2k$). We can think about it as cutting one card several times, so we do not need to consider it a new operation, but a repetition of the more basic operation of cutting one.

(ii) The flip.

The flip switches cards 1 and 2, *while also reversing them* and leaves the rest of the cards unchanged.

So the cutting operation acts on all the cards in the same way, but the flip does different things to different cards. It seems to mess up any order the cards may have had, but clearly *something* of the order must be preserved, since we are able to make all cards of the same color face the same way in the end!

What may that "something" be related to? Well, we may want to pay attention to the color and orientation of the card, and since the last step involves flipping all *even* cards, perhaps also to the card's position. Note, however, that the *color* of the card is just the *parity of its position* at the start of the trick - all black cards are in even positions, and all red cards are odd positions. So the color property is encoded in the (initial) position property. An abundance of even-odd dichotomies in the trick may also indicate that we really only need to track the *parity* of the card's position.

So I suggest that we think of every card at any given moment as being either "odd" or "even" (if it's in the odd or even position in the deck respectively) and either "up" or "down" (if it is face up or face down in the deck). Those who are into programming, may think of a card-object as having two attributes - position, and orientation, each with two possible values.

At the start of the trick, all cards are facing up, so that there are two types of cards: (odd, up) - those are the red ones; and (even, up) - those are the **black** ones.

A cut changes the position of every card. So all odd cards become even and vice versa. They are changing, but they are changing in unison. The flip either leaves the card alone, or (if the card is one of the two being flipped) changes *both* the orientation and the position. Aha! If we look at the *total number of attribute changes*, the position change and the orientation change will "cancell out". I think we are now ready to see how the 'Red and black, up and down" trick works!

The argument.

To make things a bit clearer, lets look at the case when while performing the trick we cut an even number of cards (total, for all cuts performed).

What kind of cards can we get after flips and cuts?

Well, a **black** card that started (even, up) will have an even number of changes - and so will end up either (even, up) or (odd, down). The final step in the trick - flipping all even cards - will make it (even, down) or (odd, down). So we don't know if, at the end of the trick, it is odd or even, but we know that it is always facing down!

Similarly, a red card that started (odd, up) will end up after the cuts and flips as either (odd, up) or (even, down), and at the end of the trick as (odd, up) or (even, up), that is, simply, face up.

So in the end, all cards that started at odd positions are facing up, and all **cards** that started in the even positions are facing down, just as we saw in the trick!

Note that the case where we cut a odd total number of cards would produce the opposite effect - you may find it fun to check this yourself.

Back to the Baby Hummer.

If you now go back to the Baby Hummer trick we started with, you will see that it's based on the "Red and black, up and down". Indeed, the steps 4 through 14 simply reproduce the "Red and black, up and down" for four cards (the four cards do not have to be arranged in an alternating pattern by color, but that is of noc consequence).

What do the remaining steps, steps 1-3 do?

If you denote the cards you start with as 1, 2,3, 4, with 4 on the bottom (and hence the card you memorize), then the first three moves flip over the card 2. We then run the "Red and black, up and down" on the resulting deck. If we didn't pre-flip 2, this would result in 1 and 3 facing the opposite way from 2 and 4. But 2 was pre-flipped, so now it "aligns itself" with 1 and 3 - leaving 4 as the oddball! And this is it, the mystery is explained.