Fitch Four Glory

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In 1950, mathematician and magician William Fitch Cheney Jnr. published a superb two-person mathematical card trick, which continues to baffle audiences today. It involves the use of a confederate who is very familiar with the details in advance. Both the original trick, and the more recent variation considered below, work just as well with large groups of onlookers as small ones. (The Fitch Cheney version is still in print, in Math Miracles by W. Wallace Lee, Micky Hades International, Box 476, Calgary, Alberta, Canada. Martin Gardner mentioned it in passing in several of his books in the 1950s and 1960s.)

The tricks are guaranteed 100% mathematical—though you may choose to dress them up a little, for instance as mind reading tricks. They will stump all but the most sophisticated onlookers; a general audience will be convinced that you are using some sort of body language or verbal signaling. To allay any such suspicions, one can use email, telephone, an innocent go-between, or some other form of impersonal communication. As originally published, this trick was intended to be carried out over the telephone.

The basic Fitch Cheney trick goes like this: Five cards are randomly picked from the deck and given to a mathemagician, who shows four of them (face up) to another mathemagician. This second person sees nothing else, but promptly reveals the identity of the fifth card.

This classic of mathematical magic has been revived in the past decade or two, thanks to Art Benjamin in particular, who found it buried in the Lee book in 1980. It appeared as Problem of the Week 842 on The Math Forum in late 1997. It's also been examined and championed in a generalized form—which we'll consider on a future occasion—by Hal Kierstead, Elwyn Berlekamp, Michael Kleber and others. We learnt it from Paul Zorn, over a beverage at the Joint Math Meetings in San Antonio towards the end of the last millennium. That innocent sharing of an intriguing puzzle over a quaffable brew was actually the source of our present affliction/obsession with mathematical card tricks, from which we show no signs of recovering.

There is one key point about the original trick and the variation we are about to explore: *The first mathemagician gets to decide which card to hold back for the final revelation, and hence which ones to display, and also in what order those are displayed!* 

What is not so obvious, is how seeing one of the $4! = 24$ possible arrangements of some four of the five given cards can provide the second mathematician with enough information to deduce the identity of the fifth card! The reader is *strongly urged* to figure out how this is possible before reading any further.

Less Is More

Here is a twist on the classic Fitch Cheney trick, featuring two mathemagicians, who are conveniently named Aodh and Bea.
Aodh hands out an ordinary deck of 52 cards to a spectator for shuffling, and requests that any four cards be handed back to him. Glancing at the card faces, he hands one card back, and has the spectator show it around for all to see and note, before shuffling it back into the deck. Aodh then places the other three cards in a neat row on the table, some face-up, some face-down. Bea enters the room for the first time, and surveys the scene. She promptly rummages through the deck and pulls out the card that was noted and lost.

Can you identify the lost card, guided only by the row displayed here? (It turns out that it's the 7♠.)

Even if you are familiar with the workings of the basic Fitch Cheney trick, it's rather surprising that Bea can pull this off every time—without fail—especially in the case where the three lined-up cards are face-down! Surely seeing three cards, some or all of which are face-down, is less enlightening than seeing them face-up? Actually, it can be viewed as more revealing.

This variation of the Cheney trick dates from early 1999, and first made it to print in *All You Need Is Cards*, by the reclusive Brain Epstein, published in January 2002 in *A Puzzlers' Tribute - A Feast for the Mind* (a tribute to Martin Gardner). The approach taken there is explored below, as it was in the current author's *Fitch Cheney's Five Card Trick* in MAA's *Math Horizons*, 10, February 2003, with Epstein's kind permission.

**It's as easy as Alpha, Beta, Gamma**

We'll describe one way of performing this trick. The two mathemagicians must agree on some mathematical conventions which will aid in the all-important silent communication stage later—note that the cards will do the talking!

Setting A♠ aside for now, we partition the rest of the deck into three "supersuits" of 17 cards each. Each of these consists of one of the usual suits ♠, ♦, or ♥, supplemented with four ♠'s. Specifically, Supersuit Alpha is A♠, ..., K♠, 2♠, 3♠, 4♠, 5♠; Supersuit Beta is A♦, ..., K♦, 6♠, 7♠, 8♠, 9♠; and Supersuit Gamma is A♥, ..., K♥, 10♠, J♠, Q♠, K♠.

When Aodh gets the four cards, he looks at their faces quickly. If one of them cards is A♠, then this is the card which he has noted and shuffled back into the deck. The other cards are then placed face-down, and Bea, upon seeing this, pulls the A♠ out of the deck without hesitation.

Otherwise, we note that (at least) two of the four cards must be from the same supersuit. Without loss of generality, we may assume that there are two cards from supersuit Alpha. One is noted and lost in the deck, and by placing the remaining three cards on the table in some particular fashion, the lost card's identity is secretly communicated to Bea. (Don't allow get distracted by the possibility of two suit matches, or a triple or quadruple suit match; it suffices to focus on a single match!)

Now comes the sneaky part: care must be exercised in selecting the card to hand back. Each supersuit contains 17 cards, which we can imagine arranged clockwise in a circle like a 17 hour clock, with the last of the supplemented ♠'s being followed by the appropriate Ace, 2, 3, etc.. Given any two cards in this circle, there is one (and only one) of them -- which we refer to as the higher (or target) card -- lying between one
and eight positions further around the circle (measured clockwise) than the other. There simply isn't enough room for the two cards to be nine or more positions apart in both directions! This target card is the one which is handed back to be noted and lost in the deck. We refer to the other card in this match as the lower (or base) card.

For instance, if the two cards are $4\spadesuit$, $Q\spadesuit$, then since the second is 8 past the first, the $Q\spadesuit$ is considered to be the target card, and it's the one lost and later identified by Bea. If the two cards are $6\spadesuit$, $4\spadesuit$, then since the first card is 7 past the second, the $6\spadesuit$ is the target card.

In general, Aodh gives back the target (higher) card from some supersuit, whose numerical value is $k$ past the base (lower) card, $k$ being between 1 and 8 inclusive. This base card is used, along with the other two from the four given, to let Bea know what the target card is. In the convention explained below, at least one card will be face-up, and we use the first such face-up card (reading from left to right) to communicate the supersuit of the target card -- as well as the base for the count of $k$. It only remains to communicate $k$.

Let's try a modification of the usual binary code with the face-up (U) and face down (D) cards, to indicate what $k$ is, remembering that DDD is off limits here, as that must be strictly reserved to communicate that the target card is $A\spadesuit$. We agree that DDU denotes that $k$ is 1, DUD denotes that $k$ is 2, DUU denotes that $k$ is 3, UDD denotes that $k$ is 4, UDU denotes that $k$ is 5, and UUD denotes that $k$ is 6. That leaves UUU, which we will need for both 7 and 8.

If we reserve the first U position to communicate the supersuit, as before, then there are two ways to position the other two face-up cards: Low-High (to convey $k = 7$) or High-Low (for $k = 8$), with respect to some total ordering of the deck (no circular wrap-around this time!). For this total ordering we agree to use the supersuits Alpha, Beta, Gamma, lined up like that.

Michael Trick at Carnegie Mellon kindly put together a website which illustrates this, er, trick in action. It uses the conventions established in the Epstein article, which differ a bit from what has just been suggested.

**Example 1**

Suppose Aodh is handed: $A\spadesuit$, $Q\spadesuit$, $2\heartsuit$, $7\spadesuit$.

Here the $Q\spadesuit$ and $7\spadesuit$ provide a supersuit match, with the latter card being three past the latter card in the Beta circle. So Aodh will hand back the $7\spadesuit$ and display the remaining cards so as to convey to Bea that: $Q\spadesuit$ is the base card, and the value of the target card is three past that, within the appropriate supersuit. Bea needs to see three cards in a DUU display, with the $Q\spadesuit$, in the middle. Aodh has two choices: he can display the row shown earlier, or this one:

![Image of cards]

These cards also point to the fact that the target card is $7\spadesuit$.

Bea, upon seeing that, only pays attention to the fact that it's DUU, and that the $Q\spadesuit$ is in the middle. The value of the last card actually plays no role for her! So whether she sees this arrangement, or the one shown
earlier, she merely notes that DUU corresponds to three, and hence cycles three cards past the base card in the Beta supersuit, to deduce that the target card must be the 7♠.

**Example 2**

Here’s a second example. Suppose Aodh is handed: 6♣, 8♥, 3♠, 7♠.

Here the 6♣ and 3♠ provide a supersuit match, with the former card being eight past the latter card in the Alpha circle. So Aodh will hand back the 6♣ to be noted and shuffled back in, and display the remaining cards so as to convey to Bea that the 3♠ is the base card, and also that the value of the target card within the appropriate supersuit is eight past that. Bea needs to see three cards in a UUU display, in other words all three face-up, with the 3♠ on the left, and the other two in reverse order, relative to the Alpha/Beta/Gamma order, in order to convey "Add 8" (as opposed to 7). Hence Aodh lays out the cards as:

![Card Display](image)

This display identifies the target card as the 6♣.

Upon seeing this, Bea interprets the first card in the UUU display (is in the Alpha supersuit) as the base card, and the fact that the second and third cards are in reverse order as an indication that she should go eight past 3♠. Hence she knows that the target card must be the 6♣.

**A Clopen Question Or Two**

There is a totally different direction in which the basic Fitch Cheney trick has been generalized, namely to larger decks, which is maximally efficient (unlike the above). We'll have more to say about this on a future occasion.

We've considered a mixture of face-up and face-down cards for a reason: it is well known that the basic Cheney trick—where a row of face-up cards points to the identity of one remaining face-down card—cannot be done starting with just four cards. After all, given any four cards there are at most 4! = 24 things one can do with them: four ways to select one to have noted and lost, and 3! = 6 ways to arrange the remaining three. Even allowing for the extra bit of information that Aodh can surreptitiously communicate by laying those cards—in full view of Bea—from left to right, or right to left, that only allows for 2 x 24 = 48 possibilities. Bearing in mind that three card faces would be face-up, and hence couldn't be contenders for the target card—it seems that at the very most, a deck of size 48 + 3 = 51 could be accommodated. And that assumes that we could find a strategy that took full advantage of the possibilities just enumerated. On the other hand, our alphabetagammical 52 card presentation is clearly not optimal, which leads one to wonder: just how much wriggle room is there?

We leave you with a terrific question posed by Derek "Octonion" Smith of Lafayette College:

> Can the original Fitch Cheney trick, in which five cards are randomly chosen, and four are lined up face-up to point to the fifth, be done if the five cards are drawn from two (identical) decks of 52 cards?