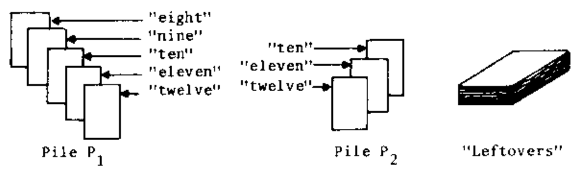
UMAP

UNIT 560

MODULES AND MONOGRAPHS IN UNDERGRADUATE
MATHEMATICS AND ITS APPLICATIONS PROJECT

SOME CARD TRICKS:
ALGEBRA IN DISGUISE

by Peter A. Lindstrom



$$\begin{pmatrix}\text{number of cards}\\ \text{in } P_1\end{pmatrix} + \begin{pmatrix}\text{number of cards}\\ \text{in } P_2\end{pmatrix} + \begin{pmatrix}\text{number of}\\ \text{left-overs}\end{pmatrix} = 15$$

$$5 + 3 + \begin{pmatrix}\text{number of}\\ \text{left-overs}\end{pmatrix} = 15$$

$$\begin{pmatrix}\text{number of}\\ \text{left-overs}\end{pmatrix} = 7.$$

APPLICATIONS OF ELEMENTARY ALGEBRA

-- RECREATIONAL MATHEMATICS

Intermodular Description Sheet: UMAP Unit 560

Title: SOME CARD TRICKS: ALGEBRA IN DISGUISE

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Classification: APPL ELEM ALG/RECREATIONAL MATH

Target Audience: General mathematics courses in high school and college.

Abstract: Many card tricks are counting problems solved by elementary algebra. Each card trick (word problem) is transformed to a linear equation in several unknowns. Solving the equation for certain variables will produce a rule for "solving" the card trick. Students develop rules for solving card tricks by elementary algebra, and show applications of elementary algebra to recreational mathematics.

Prerequisites: Transform word problems to linear equations in several variables and solve for one (or more) of the variables of a linear equation in several unknowns.

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TABLE OF CONTENTS

MODULES AND MONOGRAPHS IN UNDERGRADUATE MATHEMATICS AND ITS APPLICATIONS PROJECT (UMAP)

The goal of UMAP is to develop, through a community of users and developers, a system of instructional modules in undergraduate mathematics and its applications that may be used to supplement existing courses and from which complete courses may eventually be built.

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This module was developed under the auspices of the UMAP Precalculus Panel whose members are: Peter A. Lindstrom, Chair, Genesee Community College; Charles Biles, Humboldt State University; Marilyn Gilchrist, North Lake College; and Roland H. Lamberson, Humboldt State University. The Project would like to thank Calvin J. Holt, Jr. of Paul D. Camp Community College, Anthony Giangrasso of LaGuardia Community College - CUNY, Richard Montgomery of Southern Oregon State College, and one anonymous reviewer for their reviews, and all others who assisted in the production of this unit.

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1. INTRODUCTION

Magicians entertain us by performing tricks that seem impossible. Dollar bills are plucked out of thin air, doves are lifted from scarves, obliging assistants are sawn in half. Tricks like these are achieved by illusion and slight of hand. But other tricks, among them a large number of seemingly impossible card tricks, depend on elementary algebra rather than on deception. In this unit, we show how some of these card tricks work.

2. CARD TRICK #1

This trick uses a standard deck of 52 cards. The cards are given the following numerical values: each ace counts as "1," twos through tens count as their face value, and each jack, queen, and king counts as "10." Out of your sight, someone forms separate piles of cards in the following manner: he takes the first card from the deck, notes its numerical value and places it face down. Silently calling this card by its numerical value, he counts out additional cards (all face down) from the deck until the count reaches "12." For example, if the first card was a seven, he would silently count "seven," "eight," "nine," "ten," "eleven" and "twelve," thus forming a pile of six cards whose bottom card is seven and whose top card is assigned the value of "twelve." Other piles are formed in the same manner until the deck is exhausted or there are not enough cards to form another pile.

Now comes the trick: returning to the scene, you announce the sum of the numerical values of the bottom cards. To do this, use the following rule: Multiply the number of piles by 13, subtract 52, and add the number of left-over cards. This rule yields the sum that you announced. For example, if you return to the scene to find five piles of "counted" cards and a pile of three "left-overs," then the sum of the numerical values of the bottom cards is $(13 \cdot 5) - 52 + 3 = 16$. Before continuing, spend a few minutes with a deck of cards; make your own piles and convince yourself that the rule always seems to work.

The rule for this card trick is "solved" by elementary counting techniques and algebra. Before we look at its solution, let's develop two basic principles that are used in all of the card tricks of this module.

3. TWO BASIC PRINCIPLES

The "Wizard of Id" cartoon in Figure 1 illustrates the first of these basic principles.

> BP1 (Basic Principle 1): The sum of the numbers of cards in different piles equals the total number of cards in the deck.

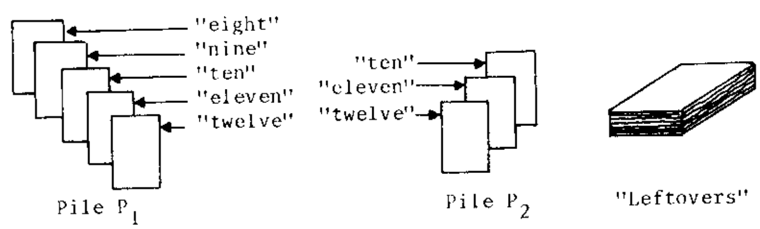
Figure 1. Reprinted by permission of Johnny Hart and Field Enterprises, Inc.

If there are 52 cards in the deck and Wiz has one of them, then

(the number of cards held by Sir Rodney) + 1 = 52.

Although this illustration is a rather trivial use of BP1, it does tell us that Rodney holds 51 cards.

To further illustrate how BP1 can be used, suppose that you have a deck of 15 cards and you form two piles, P_1 and P_2, counting values as in Card Trick #1. Assuming that the bottom card of P_1 is an 8 and the bottom card of P_2 is a king, then there are 5 cards in P_1 and 3 cards in P_2, as shown in Figure 2.



(number of cards in P_1) + (number of cards in P_2) + (number of left-overs) = 15

5 + 3 + (number of left-overs) = 15

(number of left-overs) = 7.

Figure 2.

Although Figure 2 illustrates BP1, it can also be used to introduce the second basic principle, which relates the number of cards in a pile to the values of its top and bottom cards. The two following exercises introduce this principle.

Exercise 1. Of the 5 cards of P_1 of Figure 2, the bottom card has a value of "8" and the top card has been assigned a value of "12."
(a) Explain why 12 - 8 does not tell you the number of cards in P_1.
(b) Explain why $12 - 10 \neq$ the number of cards in P_2.

(Before doing Exercise 2, be sure that you understand Exercise 1.)

Exercise 2. Suppose that in forming a pile of cards, counting values as in Card Trick #1, the bottom card has a value of B and the top card has been assigned a value of T. Develop a formula for the number of cards in the pile in terms of B and T.

The result of Exercise 2 will be known as Basic Principle 2.

> BP2(Basic Principle 2): In forming a pile of cards whose bottom card has a value of B and whose top card is assigned the value of T, the number of cards in the pile is $(T + 1) - B$.

The following exercise uses both BP1 and BP2 for its solution.

Exercise 3: Suppose that you have a deck of 36 cards. You form two piles P_1 and P_2 (as in Card Trick #1), where a 5 is the bottom card of P_1 and a jack is the bottom card of P_2. In forming these piles, suppose that you stop counting with "10" for P_1 and "14" for P_2. Determine the number of cards left over after the two piles are formed.

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4. SOLUTION TO CARD TRICK #1

By algebraic methods and the two principles discussed in the previous section, you are now prepared to set up a formula and a rule for the solution of Card Trick #1.

Let B_1 = the numerical value of the bottom card of the first pile.

Let B_2 = the numerical value of the bottom card of the second pile.

Suppose that k piles are formed, so that for the last pile, you let

B_k = the numerical value of the bottom card of the k^{th} pile.

By BP2, there are

$$(12 + 1) - B_1 = (13 - B_1) \text{ cards in the first pile,}$$
$$(13 - B_2) \text{ cards in the second pile,}$$
$$\vdots$$
$$(13 - B_k) \text{ cards in the } k^{th} \text{ pile.}$$

Suppose that there are D cards left over after the k piles have been formed. By BP1, we then have

$$(1)\quad \underbrace{(13-B_1) + (13-B_2) + \ldots + (13-B_k)}_{\text{number of cards in the k piles}} + \underbrace{D}_{\substack{\text{number}\\ \text{of cards}\\ \text{left over}}} = \underbrace{52.}_{\substack{\text{number of}\\ \text{cards in}\\ \text{the deck}}}$$

Since you want to announce what the sum of the numerical values of the bottom cards of all of the piles is, you want to determine from (1), the value of

$$(B_1 + B_2 + \ldots + B_k).$$

Where we group the terms in (1), we find that

$$\underbrace{(13 + 13 + \ldots + 13)}_{\text{k values of 13}} - B_1 - B_2 - \ldots - B_k + D = 52$$

$$13 \cdot k - (B_1 + B_2 + \ldots + B_k) + D = 52$$

$$(2)\quad 13 \cdot k - 52 + D = (B_1 + B_2 + \ldots + B_k).$$

Since the right side of (2) is the number you seek, you can obtain it from the left hand side of (2). Thus, to find your number you need to know only the number of piles, k, and the number of cards, D, that are left over. Hence, you (and algebra!) have derived the stated rule for the solution of Card Trick #1.

Exercise 4. (Card Trick #2): Suppose that you have a deck of 42 cards. You form three piles of cards counting values as in Card Trick #1, except that you count out cards until you reach "13" for each pile. Develop a formula, and then a rule, that will tell you the sum of the numerical values of the bottom cards of the three piles if you know how many cards are left over.

Exercise 5. (Card Trick #3): You have a deck of 39 cards. You form only three piles of cards counting values as in Card Trick #1 by the following methods:

First Pile: Count out cards until you reach "10."
Second Pile: Count out cards until you reach "12."
Third Pile: Count out cards until you reach "14."

Develop a formula, and then a rule, that will tell you the numerical value of the bottom card of one of the piles if you are told the sum of the numerical values of the bottom cards of the other two piles and how many cards are left over.

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5. CARD TRICK #4

This trick uses a standard deck of 52 cards. Here though, the cards have the following numerical values: an ace counts as "1," twos through tens count as their face values, a jack counts as "11," a queen counts as "12," and a king counts as "13." Again, piles of cards are formed out of your sight by the dealer but in the following manner: the top card of the deck is dealt face up, its numerical value is noted, and additional cards are dealt (also face up) and counted until "13" is reached. The process is continued until at least three piles are formed; the undealt cards (if any) are retained by the dealer. The dealer then turns all the piles face down and selects any three piles to finish the trick; the remaining piles are combined with the undealt cards. The dealer then turns over the top cards of two of the piles.

Now the rule for the trick: you return to the scene and take the remaining cards from the dealer (all cards that are not in the three piles) and deal out a number of cards equal to 10 more than the sum of the values of the two exposed cards. The number of cards you have left is the numerical value of the unexposed top card of the third pile!

To analyze this trick, you need to consider only the three piles which are eventually turned face down. Let T_1, T_2 and T_3 be the numerical values of the top cards of the first, second and third piles respectively, so that the number of cards in these piles is $(14 - T_1)$, $(14 - T_2)$ and $(14 - T_3)$. If D is the number of cards remaining, then

$$(14 - T_1) + (14 - T_2) + (14 - T_3) + D = 52$$

$$42 + D - (T_1 + T_2 + T_3) = 52$$

$$D - (T_1 + T_2 + T_3) = 10. \qquad (3)$$

Suppose that you know the numerical value of the top cards of the first and second piles, so that you can determine $(T_1 + T_2)$. Solving (3) for T_3 you obtain

$$T_3 = D - (10 + T_1 + T_2).$$

Hence, the numerical value of the unexposed top card of the third pile, T_3, is obtained by subtracting 10 more than the sum of the values of the two exposed cards from the number of remaining cards. In other words, if you deal out (take away) from the remaining cards a number of cards equal to 10 more than the sum of the values of the two exposed cards, the number of cards that you have left is the numerical value of the unexposed top card of the third pile.

Exercise 6. (Card Trick #5): Suppose that in Card Trick #4, at least four piles are formed and exactly four piles are turned face down. Develop a formula and then a rule that will tell you the numerical value of the top card of one of the piles if you know the number of remaining cards and the top card of the other three piles.

Exercise 7. (Card Trick #6): Divide a standard deck of 52 cards into two separate face-down piles of 26 cards each, calling one pile A and the other Z. The numerical values of the cards are identical to those assigned to Card Trick #1: Each ace counts as "1," twos through tens count as their face value, and each face card counts as "10." From pile A, count down to the 7^{th} card from the top and show it to someone but do not look at it yourself. Then return the 26 cards in pile A to their original face-down order. Using pile Z, deal the cards face up to form a pile Z_1 of cards in the following manner: Note the assigned numerical value of the first card and then deal out additional cards (all face-up) until "10" is reached. Two more face-up piles, Z_2 and Z_3 are formed in the same manner. When these three piles have been formed, place the remaining cards from pile Z face-down on the top of pile A to form a pile C (all face down). Next turn piles Z_1, Z_2 and Z_3 over (so all of these cards become face-down), expose the top card of each pile, and add up the numerical values of these three exposed cards. Pick up pile C and count down a number of cards equal to the sum of the numerical value of the three exposed cards. Which card is it? Why does this trick work?

Exercise 8. Card Trick #6 does not always work. Investigate Formula (4) (of the Solution to Exercise 7) and show why the trick does not always work.

Exercise 9. (Optional): Now that you have seen a variety of card tricks and algebraic solutions of each, develop your own card trick and verify its solution algebraically.

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6. CONCLUSION

There is nothing magical or mysterious about the card tricks presented in this module. The solutions do not involve any illusions or "sleight of hand;" counting techniques and algebra yield the solutions for you.

7. MODEL EXAM

1. (Card Trick #7): Suppose that, out of your sight, someone forms (from a deck of at least 36 cards) three piles of cards counting values as in Card Trick #1, by the following methods:

   First Pile: Count cards until you reach "10."
   Second Pile: Count cards until you reach "11."
   Third Pile: Count cards until you reach "12."

   Develop a formula, and then a rule, that will tell you the number of cards in the deck in terms of the number of cards left over and the numerical values of the bottom cards of the three piles.

2. (Card Trick #8): You have a standard deck of 52 cards and form four piles of cards counting values as in Card Trick #1, by the following methods:

   First Pile: Count cards until you reach "10."
   Second Pile: Count cards until you reach "11."
   Third Pile: Count cards until you reach "12."
   Fourth Pile: Count cards until you reach "13."

   Develop a formula and then a rule that will tell you the number of cards left over (after the four piles have been formed) in terms of the numerical values of the bottom cards of the four piles.

3. (Card Trick #9): You have a standard deck of 52 cards and form four piles of cards counting values as in Card Trick #1, by the following methods:

First Pile: Count cards until you reach "10."
Second Pile: Count cards until you reach "10."
Third Pile: Count cards until you reach "11."
Fourth Pile: Count cards until you reach "11."

Develop a formula and then a rule that will tell you the sum of the bottom cards of the first two piles in terms of the number of cards left over and the sum of the bottom cards of the last two piles.

8. SOLUTIONS TO EXERCISES

1. When 8 is subtracted from 12, the result does not account for all of the cards. The expression 12 - 8 accounts for the number of cards after the card whose value is 8 and up through the card whose value is 12. To include the card whose value is 8, you must add 1 to the difference. Hence, $(12 - 8) + 1 = 5$, the number of cards in P_1. In the same manner, $(12 - 10) + 1 = 3$, the number of cards in P_2.

2. The difference, T - B, accounts for the number of cards after the card whose value is B and up through the card whose value is T. To include the card whose value is B, you must add 1 to the difference. Hence, $(T - B) + 1 = (T + 1) - B$ is the number of cards in the pile of cards whose bottom card has a value of B and whose top card is assigned the value of T.

3. Let D be the number of cards left over after the two piles are formed. By BP2,

   $(10 + 1) - 5$ cards are in pile P_1,

   and

   $(14 + 1) - 10$ cards are in pile P_2.

   By BP1,

$$\underbrace{((10 + 1) - 5)}_{\text{number of cards in pile } P_1} + \underbrace{((14 + 1) - 10)}_{\text{number of cards in Pile } P_2} + \underbrace{D}_{\text{number of cards left over}} = \underbrace{36}_{\text{number of cards in the deck}}$$

$$6 + 5 + D = 36$$

$$D = 25.$$

   Hence, there are 25 cards left over after the two piles are formed.

4. Let B_1, B_2, and B_3 be the numerical values of the bottom cards of 1^{st}, 2^{nd} and 3^{rd} piles.

By BP2,

$(14 - B_1)$ is the number of cards in 1st pile.

$(14 - B_2)$ is the number of cards in 2nd pile.

$(14 - B_3)$ is the number of cards in 3rd pile.

If there are D cards left over, then by BP1

$$(14 - B_1) + (14 - B_2) + (14 - B_3) + D = 42$$

$$3 \cdot 14 - B_1 - B_2 - B_3 + D = 42$$

$$42 - (B_1 + B_2 + B_3) + D = 42$$

$$-(B_1 + B_2 + B_3) + D = 0$$

$$D = (B_1 + B_2 + B_3).$$

<u>Rule</u>: The sum of the numerical values of the bottom cards of the three piles is the same as the number of cards left over.

5. $\underbrace{(11 - B_1)}_{\text{number of cards in first pile}} + \underbrace{(13 - B_2)}_{\text{number of cards in second pile}} + \underbrace{(15 - B_3)}_{\text{number of cards in third pile}} + \underbrace{D}_{\text{number of cards left over}} = \underbrace{39}_{\text{total number of cards}}$

$$11 + 13 + 15 - B_1 - B_2 - B_3 + D = 39$$

$$(11 + 13 + 15) - (B_1 + B_2 + B_3) + D = 39$$

$$39 - (B_1 + B_2 + B_3) + D = 39$$

$$-(B_1 + B_2 + B_3) + D = 0$$

$$D = B_1 + B_2 + B_3.$$

Suppose that you are told the numerical values of the bottom cards of the first and third piles, B_1 and B_3. Then,

$$B_2 = D - (B_1 + B_3).$$

<u>Rule</u>: To find the numerical value of the bottom card of one of the piles, subtract the sum of numerical values of the bottom cards of the other two piles from the number of cards left over.

6. Let T_1, T_2, T_3, and T_4 be the numerical values of the top cards of the first, second, third and fourth piles, and let D be the number of cards left over. Then,

$$(14 - T_1) + (14 - T_2) + (14 - T_3) + (14 - T_4) + D = 52$$

$$56 + D - (T_1 + T_2 + T_3 + T_4) = 52$$

$$D - (T_1 + T_2 + T_3 + T_4) = -4.$$

Suppose that you know the numerical value of the top cards of, say, the first, third, and fourth piles, so that you can determine $(T_1 + T_3 + T_4)$. Then,

$$D - T_2 = (T_1 + T_3 + T_4) - 4$$

$$T_2 = D - ((T_1 + T_3 + T_4) - 4),$$

which is a formula for determining T_2.

Rule: To determine the numerical value of the unexposed top card, subtract 4 from the total of the numerical values of the three exposed cards and count out that many cards. The number of cards that are left then is the numerical value of the last unexposed top card.

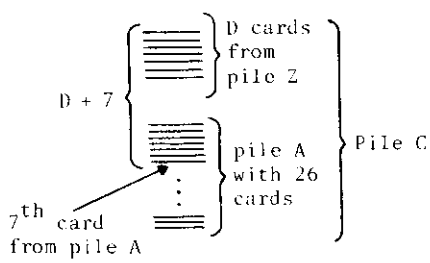
7. Let T_1, T_2, and T_3 be the numerical values of the top cards of the three piles, after these piles have been formed and turned face down. Then $(11 - T_1)$, $(11 - T_2)$ and $(11 - T_3)$ are the number of cards in the first, second, and third piles. Since these piles were formed from pile Z,

(4) $(11 - T_1) + (11 - T_2) + (11 - T_3) + D = 26,$

where D is the number of cards left over and used to form pile C. Since $(T_1 + T_2 + T_3)$ is the sum of the numerical values of the three exposed cards, let's solve (4) for $(T_1 + T_2 + T_3)$. Then

$$33 - (T_1 + T_2 + T_3) + D = 26$$

(5) $$7 + D = T_1 + T_2 + T_3.$$



Since the number of cards counted out from pile C is $T_1 + T_2 + T_3$ in (5), this is the same as 7 + D, the sum of 7^{th} card down from pile A and the number of cards left over from pile Z.

8. If the three top exposed cards are all aces, then

$T_1 = T_2 = T_3 = 1$, so that (4) becomes

$$(11 - 1) + (11 - 1) + (11 - 1) + D = 26$$

$$D = -4.$$

But D cannot be a negative number. In general, there are other values of the three top exposed cards for which the trick will not work. From (5) we have

$$D = (T_1 + T_2 + T_3) - 7 \ .$$

Since D cannot be negative,

$$(T_1 + T_2 + T_3) \geq 7.$$

Hence, the sum of the three top exposed cards has to be at least 7.

9. SOLUTIONS TO THE MODEL EXAM

1. By BP2 and BP1,

$$(11 - B_1) + (12 - B_2) + (13 - B_3) + D = X,$$

where D is the number of cards left over and X is the number of cards in the deck. In other words,

$$(36 + D) - (B_1 + B_2 + B_3) = X.$$

<u>Rule</u>: To determine the number of cards in the deck, subtract the sum of the numerical values of the three bottom cards from the sum of 36 and the number of cards left over.

2. By BP2 and BP1,

$$(11 - B_1) + (12 - B_2) + (13 - B_3) + (14 - B_4) + D = 52$$

$$50 - (B_1 + B_2 + B_3 + B_4) + D = 52$$

$$D = (B_1 + B_2 + B_3 + B_4) + 2.$$

<u>Rule</u>: To determine the number of cards left over, add 2 to the sum of the numerical values of the four bottom cards.

3. By BP2 and BP1,

$$(11 - B_1) + (11 - B_2) + (12 - B_3) + (12 - B_4) + D = 52$$

$$46 - (B_1 + B_2) - (B_3 + B_4) + D = 52$$

$$(B_1 + B_2) = D - (6 + B_3 + B_4).$$

<u>Rule</u>: To determine the sum of the bottom cards of the first two piles, subtract from the number of cards left over the sum 6 and the values of the bottom cards of the last two piles.