The Twenty-Seven Card Trick
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Before discovering the recurrences for $W(N, C, n, i)$, I simulated the problem using random shuffles of a deck of 52 cards on a computer. In 71 166 shuffles, a given pair occurred 34 486 times, giving Prob(pair) $\simeq 0{\cdot}484585$, which agrees well with the exact result above.

I made some investigations into a possible recurrence for $W(N, C, n)$ and into the behaviour of $E(n) = \sum n\, W(N, C, n) \Big/ \binom{N}{C}$, but neither seems to give any simple results.

*References*

Robert Harbin, *Waddingtons Family Card Games*, Prediction, p 178. Elm Tree Books (Hamish Hamilton), London (1972).

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## The twenty-seven card trick

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Mathematical magic, properly used, can play an important role in the mathematics classroom. Students need to be motivated to study mathematics (or any subject), and unusual and unexpected results, though sometimes frivolous, often provide more effective motivation than the utilitarian ideas of real world applications. Such an idea was discussed by Anderson [1] in his article on the twenty-one card trick. Here we describe a generalisation of Anderson's idea based on arithmetic in base $b$ for any integer $b > 1$. However, for ease in explication, we describe the trick in base 3 terminology.

### *The trick*

Ask your class to select an integer $n$ between 1 and 27 inclusive, and to tell you the integer chosen. Also, ask the class secretly to select one of 27 cards dealt from an ordinary deck. The trick is to distribute the 27 cards face up into 3 columns of 9 cards each and to pick them up in such a way that, after repeating the process three times, the selected card is in the chosen $n$th position in the deck. In fact, the trick can be made even more spectacular if you have a randomly chosen member of the class manipulate the cards while you stand across the room where the cards cannot possibly be seen. Of course, dressing up the show with statements about your psychic or occult powers while exhorting the students to concentrate deeply on the selected card only helps to heighten the effect when the selected card finally turns up in the proper position.

In detail, the trick is performed as follows.

1. Lay out the 27 cards face up in three columns of 9 each as shown in Fig. 1.

| | | |
|---|---|---|
| 0 | 1 | 2 |
| 3 | 4 | 5 |
| 6 | 7 | 8 |
| 9 | 10 | 11 |
| 12 | 13 | 14 |
| 15 | 16 | 17 |
| 18 | 19 | 20 |
| 21 | 22 | 23 |
| 24 | 25 | 26 |

FIGURE 1.

Note that the numbers in the figure only indicate the way the cards are to be distributed and the resulting position any given card occupies after the distribution has been made. Thus, 0 might indicate the 2 of spades, 1 the 10 of diamonds, etc. Also, the numbering is from 0 to 26 rather than from 1 to 27 simply to facilitate the proof given below.

2. Doing some quick mental arithmetic (preferably while laying out the cards as above), subtract 1 from the number the class selected and express the result in base 3 notation. Thus, if they select $n$, write $n-1$ in the form

$$n - 1 = 3^2 a + 3b + c = abc_3$$

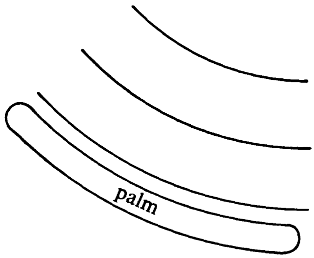
with each of $a$, $b$, and $c$ equal to 0, 1, or 2 as appropriate.

3. After the cards are distributed, ask the students to indicate the column in which their chosen card appears.

4. Now pick up the cards by columns and place them **face up** in your palm. But this must be done in a very special way. Think of the three possible positions for placing the columns in your palm. A column can be placed in

- the 0 position—next to your palm,
- the 1 position—in the middle,
- the 2 position—furthest away from your palm

as indicated in the following diagram.




Recalling that $n - 1 = abc_3$, this first time the cards are picked up, make sure that the column containing the card selected by the class is placed in the $c$ position in your palm.

5. Now, turning the assebled cards **face down** in your hand, redistribute them as in Fig. 1, and again ask the students to indicate the column in which their chosen card appears.

6. Again pick up the cards by columns but this time place the column with the chosen card in the $b$ position in your palm.

7. Finally, repeat step 5 and again pick up the cards by columns, this time placing the column with the selected card in the $a$ position in your palm.

8. All that remains now is to place the assembled deck of 27 cards **face down** in your palm. Counting out the cards from the top down, the class will be surprised and impressed when the $n$th card turned over is the card they selected!

When the trick is performed, the payoff is immediate. Students can hardly wait for an explanation, and this provides the ideal opportunity for you, the teacher, to explain that the trick is mathematical and then to review or to introduce base 3 arithmetic. Also, to be able to perform the trick well, students will have to practice and to develop the ability to do the necessary mental arithmetic quickly and easily—itself a worthwhile goal.

But this is not all. A neat "what if" question is, "Could you do this trick in another base? If so, how would it work? If not, could it be modified in some way?" Of course, in line with current emphases on problem solving and critical thinking, students should be left to work out these answers for themselves with only occasional comment and encouragement from the teacher. The fact is, however, that the trick will work in any base $b > 1$, that you use $b^r$ cards, that you must lay them out $r$ times in $b$ columns of $b^{r-1}$ cards each, that the positions in the palm correspond to the digits, 0, 1, . . ., $b - 1$ numbered from the palm out, and that after picking up the cards for the last time, the selected card appears in the pre-selected position in the deck as desired. The generalisation from base 3 to base $b$ is exact.

*An explanation for base 3*

Think of the three columns of cards as the zeroth column, the first column, and the second column, and think of the $i$th card in each column where $0 \leqslant i \leqslant 8$. Thus we count our columns and cards in each column starting with 0 instead of 1 just as we numbered the cards in the deck from 0 through 26 rather than from 1 through 27.

To show that the trick works, we first determine where the chosen card will appear on any given distribution given that it appears as the $i$th card in the $j$th column on the preceding distribution with $0 \leqslant i \leqslant 8$ and $0 \leqslant j \leqslant 2$. Suppose that when the $j$th column on the preceding distribution was placed in the palm for the present distribution, it was placed in position $k$, $0 \leqslant k \leqslant 2$, in the palm. For $k = 0$, if $i = 0$, 1, or 2, the selected card will appear as card number 0 in column number 0, 1, or 2 respectively. If $i = 3$, 4, or 5, the selected card will appear as card number 1 in column number 0, 1, or 2 respectively. And if $i = 6$, 7, or 8, the selected card will appear as card number 2 in column number 0, 1, or 2 respectively. Thus, for $k = 0$, if we divide $i$ by 3 to obtain $i = 3q + r$ with $0 \leqslant r < 3$, the desired card appears as card number $q$ in column number $r$. Similarly, we determine that, for $k = 1$ and $i = 3q + r$ with $0 \leqslant r < 3$, the desired card appears as card number $3 + q$ in column number $r$. For $k = 2$, the desired card appears as card number $2.3 + q$ in column number $r$. Finally, in summary, the desired card appears as the $3k + q$ card in column number $r$ for $i = 3q + r$, $0 \leqslant i \leqslant 8$, $0 \leqslant r \leqslant 2$ so that $0 \leqslant q < 3$ as well.

Now let $n$ be the number selected by the class and let

$$n - 1 = abc_3 = 9a + 3b + c$$

be the base 3 representation of $n - 1$ so that $a$, $b$, and $c$ are each equal to one of 0, 1, or 2. Also, suppose that the card selected by the class is the $3i + j$ card in the deck so that on the first distribution the selected card appears as the $i$th card in the $j$th column.

Pick up the cards, placing the column containing the selected card in the $c$ position in the palm. Then, by the above, the selected card will appear as the $q_1 + 3c$ card in the $r_1$ column where $i = 3q_1 + r_1$ with $0 \leqslant r_1 < 3$ and $0 \leqslant q_1 < 3$.

Pick up the cards a second time and now place the column containing the selected card in the $b$ position in the palm. Then again, on the next distribution, the selected card appears as the $q_2 + 3b$ card in the $r_2$ column where $q_1 + 3c = 3q_2 + r_2$ with $0 \leqslant r_2 < 3$. But since $0 \leqslant q_1 < 3$, this implies that $q_1 = r_2$ and $c = q_2$.

Now pick up the cards a third time and place the column with the selected card in the $a$ position in the palm. Then, if the cards were to be distributed again, the selected card would appear as the $q_3 + 3a$ card in the $r_3$ column where $q_2 + 3b = 3q_3 + r_3$ with $0 \leqslant r_3 < 3$. But since $q_2 = c$, it follows that $0 \leqslant q_2 < 3$ so that $r_3 = q_2 = c$ and $q_3 = b$; i.e., the selected card is card number

$b + 3a$ in column number $c$. But, as noted above, this implies that the selected card is card number

$$3(b + 3a) + c = 9a + 3b + c = n - 1$$

in the deck. Thus, if we turn the deck face down in the palm and count off the cards starting from 1 instead of 0, the selected card will be the $n$th card as claimed.

*Reference*

1. Oliver D. Anderson, The twenty-one card trick, *Math. Gazette* **69**, 188–191 (1985).

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# Hidden strengths of dimensional analysis

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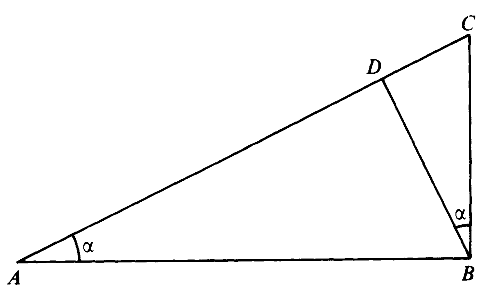
*Introduction*

Perhaps the most extensive use of dimensional analysis in applied mathematics is in the application of the *principle of similitude* to obtain equivalence between different physical systems of unequal scales occurring in widely different branches of science and engineering. However, there is a widespread misconception that dimensional analysis can only yield the functional dependence of a physical quantity on various other parameters. In this article I shall therefore demonstrate some hidden strengths of dimensional analysis.

*Pythagoras's theorem*




In the right-angled triangle $ABC$ above, it is always possible to draw a perpendicular from $B$ to $AC$ meeting it at $D$. Then obviously triangles $ABC$,