The Probability an Amazing Card Trick Is Dull
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The Ashland University student chapter of the MAA holds biweekly meetings, typically consisting of a short business meeting to plan activities that the group is sponsoring followed by a social time when the group plays mathematical games and munches on brownies. As a new feature in Fall 2002, I told the students that I would perform a new mathematical card trick at each of these meetings. My source for most of these card tricks is the delightful book [2]. One of the finest tricks described there is due to an amateur New York magician named Henry Christ. A spectator shuffles a deck of cards several times, and then the magician deals nine cards face down on a table. The spectator selects one of these cards, looks at it, and stacks the nine dealt cards on the table with the selected card on top. The magician places this stack on the bottom of the deck, and hands the deck to the spectator. The spectator is told to deal the cards out face up in a pile, counting backwards out loud from ten while dealing the cards. If at some point, the denomination of the card matches the number the spectator says, then the spectator repeats this procedure with a second pile, again counting backwards from ten. If 1 is reached without any cards matching, the spectator places the next card face down on top of the pile, and repeats this procedure with the second pile. The spectator does this until four piles are created in this manner, and then adds the numbers appearing on the cards that are face up. The spectator counts through the remaining cards of the deck to find the card in the position represented by this sum, and this card is the one that was originally selected.

The reason why this card trick works is that the final card will always be the 44th card in the deck. This can be seen by noting that any unmatched pile will contain 11 cards that are set aside. If a pile has a match with a value $i$ showing, then the match occurred after counting down $11 - i$ cards. This card with value $i$ will contribute $i$ to the sum determining how many cards in the final deck to count down, and thus, a total number of $11 - i + i = 11$ cards will be set aside for this pile.

After practicing this card trick a few times, I was ready to present it to the students at our meeting. Unfortunately, when I performed the trick at the meeting, a match did not occur in any of the four piles. This is unfortunate for two reasons. First, nothing is done with the pile of cards left in the spectator’s hands, and the magician must tell the spectator to simply turn over the card appearing face down on the final pile. Second, and more serious from the perspective of the magician, the trick seems quite simplistic when no matches occur because it is clear that you set aside 44 cards. Thus, a magician may ask, “What is the probability that no matches occur and my trick is dull?” This question falls within a class of problems known as counting permutations with restricted positions. For a more general discussion of this class of problems, see [1]. For
a more advanced treatment of the topic, see [7] or [8], and for additional applications, see [3], [4], [5], [6].

The probability of no matches

Our first step in studying this card trick problem is to note that after the nine cards are returned to the bottom of the deck, the order of the cards in the deck is one of the possible 52! permutations. Hence, for our purposes, we just need to focus on the latter part of the card trick. Let $D$ be the number of ways to shuffle the deck of cards such that there is not a 10 in the 1st, 11th, 21st, or 31st positions, there is not a 9 in the 2nd, 12th, 22nd, or 32nd positions, etc. Thus, each card from ace and ten has four positions where it cannot occur. We find $D$ by using the principle of inclusion-exclusion: Given a collection of finite sets $A_1, A_2, \ldots, A_n$, the number of elements in the union of these sets $A_1 \cup A_2 \cup \cdots \cup A_n$ is

$$|A_1 \cup A_2 \cup \cdots \cup A_n| = \sum_{1 \leq i \leq n} |A_i| - \sum_{1 \leq i < j \leq n} |A_i \cap A_j| + \cdots + (-1)^{n+1}|A_1 \cap A_2 \cap \cdots \cap A_n|.$$ 

Essentially this means to find the number of elements that appear in at least one of these sets, count the number of elements in each set, then subtract off the number of elements that appear in two sets, then add the number of elements that appear in three sets, and continue to alternate these additions and subtractions up to the number of elements that appear in all of the sets. To apply this principle to our problem, we number the forty cards with restricted positions and let $A_i$ be the set of permutations that have card numbered $i$ in one of its restricted positions. Then the number of permutations without any cards in restricted positions is

$$D = 52! - |A_1 \cup A_2 \cup \cdots \cup A_{40}|$$

$$= 52! - \left( \sum_{1 \leq i \leq 40} |A_i| - \sum_{1 \leq i < j \leq 40} |A_i \cap A_j| + \cdots - |A_1 \cap A_2 \cap \cdots \cap A_{40}| \right)$$

$$= 52! + \sum_{k=1}^{40} (-1)^k \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq 40} |A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}|.$$ 

The inner sum of this formula overcounts the number of ways to place at least $k$ cards in restricted positions. Let $r_k$ (sometimes called a rook number due to an interpretation concerning placing non-attacking rooks on a chessboard) be the number of ways of placing $k$ cards in restricted positions, ignoring the placement of any other cards. Then the remaining cards can be placed in $(52 - k)!$ ways, and we see that

$$\sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq 40} |A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}| = r_k (52 - k)!.$$

Taking $r_0 = 1$ (as is standard), we see that our formula becomes

$$D = \sum_{k=0}^{40} (-1)^k r_k (52 - k)!,$$

(1)
for which we need the numbers \( r_k \). To calculate \( r_k \), we find the generating function (called the rook polynomial) for this sequence, \( R(x) = r_0 + r_1 x + r_2 x^2 + \cdots + r_{40} x^{40} \). (Note that \( r_i \) is simply the coefficient of \( x^i \).) We do this by first finding the generating polynomial \( P(x) \) for the number of ways for having \( i \) cards for a particular denomination in a restricted position, and then observing that \( R(x) = (P(x))^{10} \).

For ease of description, we focus on placing 10's in restricted positions, but clearly the same holds for each denomination. There are \( \binom{10}{4} = 16 \) ways to place one 10 since there are \( \binom{4}{4} = 1 \) ways to choose the 10 and four positions for it. Similarly, for two 10's, there are \( \binom{4}{4} = 1 \) ways to choose the cards, and then \( 4 \cdot 3 \) ways to place them in restricted positions (with the suits in alphabetical order say, four choices for the "first" and three for the "second") for a total of \( \binom{4}{2} \cdot 4 \cdot 3 \), or 72. In the same way, we find there are 96 ways to place three 10's and 24 ways to place all four 10's in restricted positions. Therefore, (with there being just one way to have no 10's in restricted positions), \( P(x) = 1 + 16x + 72x^2 + 96x^3 + 24x^4 \).

Now let's consider the cards of two denominations, say the 10's and the 9's. We can find the number of ways of having various combinations of these eight cards in restricted positions (without regard to any others) by squaring \( P(x) = 1 + 32x + 400x^2 + 2496x^3 + 8304x^4 + 14592x^5 + 12672x^6 + 4608x^7 + 576x^8 \). For example, consider the term \( 2496x^3 \). This says there are 2496 ways of having three 9's and 10's in restricted positions, arising from no 9's and three 10's (196 ways), one 9 and two 10's (16 \cdot 72 \text{ ways}), two 9's and one 10 (72 \cdot 16 \text{ ways}), and three 9's and no 10's (96 \cdot 1 \text{ ways}). The multiplicative rule is justified since the restricted positions for different denominations are disjoint. From this line of reasoning it follows that the coefficient of \( x^4 \) in \( (P(x))^{10} \) is the number of ways of having \( k \) number cards in restricted positions, ignoring the placement of the other cards. Thus, \( R(x) = (1 + 16x + 72x^2 + 96x^3 + 24x^4)^{10} \).

We used MAPLE to expand \( R(x) \) to find the numbers \( r_k \), and then used equation (1) to get

\[
D = 32867929375495539428826782591912851876783001405845125895657881600
\approx 3.29 \times 10^{66}.
\]

Therefore, the probability that no card is in a forbidden position is \( \frac{D}{52^5} \approx 0.0407 \). We see that although it is unlikely that the magician would not get any cards to match while performing the magic trick, it will happen often enough that the magician better know how to adjust the trick. (An anonymous referee noted that we can also produce the approximate value of 0.0407 using double-precision arithmetic.)

**Exercises**

1. The game Euchre uses only 6 cards (9, 10, Jack, Queen, King, Ace) from each suit. Suppose you are at a party and you wish to demonstrate the card trick using such a deck. You plan to have the spectator create 4 piles while counting backwards Jack-10-9. How many cards should you first deal? What is the probability that no match occurs in the four piles?

2. Consider the general version of Henry Christ's trick. You have a deck of cards consisting of \( k \) suits having \( f \) face values and in the trick the spectator is to create \( p \) piles while counting backwards from \( m \) to 1. What are the possible values for \( m \)? How many cards should you first deal? Find an expression for the probability that no match occurs in the \( p \) piles.
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References


First Find a Common Numerator

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Problem. In a town in which 2/3 of the men are married to 3/5 of the women, what fraction of the adult population is married?

One method of solution. First, express both fractions with a common numerator.

Thus, in our problem, 6/9 of the men are married to 6/10 of the women.

The first six cells of the men (top row) are married to the first six cells of the women (bottom row). Clearly 12/19 of the adult population is married.

This can be generalized. If a/b of the men are married to c/d of the women, then ca/cb of the men are married to ca/da of the women. So the proportion that is married is 2ca/(cb + da).

Hmm . . . , not only did I find common numerators here, but then I combined the two fractions by dividing the sum of their numerators by the sum of their denominators. You may want to consider calling the math police . . . .