Modeling Mathematics With Playing Cards

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Over a period of more than twenty-five years **Martin Gardner** wrote *Scientific American's* column on mathematical recreations. He recently updated and expanded Silvanus Thompson's classic textbook *Calculus Made Easy*. His latest book, *Mental Magic*, is a collection for children of mathematical magic tricks.

Because playing cards have values 1 through 13 (jacks 11, queens 12, kings 13), come in two colors, four suits, and have fronts and backs, they provide wonderfully convenient models for hundreds of unusual mathematical problems involving number theory and combinatorics. What follows is a choice selection of little known examples.

One of the most surprising of card theorems is known as the Gilbreath principle after magician Norman Gilbreath who first discovered it. Arrange a deck so the colors alternate. Cut it so the bottom cards of each half are different colors, and then riffle shuffle the halves together. Take cards from the top in pairs. Amazingly, every pair will consist of a red and black card!

Here is a simple proof by induction that this must happen. Assume that the first card to fall on the table during the shuffle is black. If the next card to fall is the card directly above it in the same half, that card will be red. This places on the table a red-black pair. If the next card after the first one comes from the *other* half, it too will be red and will put a red-black pair on the table. In either case, after two cards have dropped, the bottom cards of each half will be of different colors, so the situation is exactly the same as before, and the same argument applied for the rest of the cards. No matter how careful or careless the shuffle, it will pile red-black pairs on the table.

Gilbreath's principle generalizes. Arrange the deck so the suits are in an order, say spades, hearts, clubs, diamonds, that repeats throughout the pack. Deal as many cards as you like to form a pile. This of course reverses the order of the suits. When the pile is about the same size as the remaining portion of the deck, riffle shuffle the two portions together. If you now take cards in quadruplets from the top of the shuffled pack, you will find that each set of four contains all four suits. The ultimate generalization is to shuffle together two decks, one with its cards in the reverse order of the other deck. After the shuffle, divide the 104-card pack exactly in half. Each half will be a complete deck of 52 different cards!

What mathematician David Gale has called the "non-messing-up theorem" is another whimsical result. From a shuffled deck, deal the cards face up to form a rectangle of any proportion. In each row, rearrange the cards so their values do not decrease from left to right. In other words, each card has a value higher than the one on its left, or two cards of the same value are side by side. After ordering the rows, do the same thing with the columns. This of course drastically alters the order of cards in the rows. After rearranging the columns, you may be amazed to find that the rows are still ordered!

The theorem is at least a hundred years old. You will find it proved as the answer to a problem in the *American Mathematical Monthly* (70:2, 1963, 212-13), and in a monograph by Gale and Richard Karp, published in 1971 by the operations research center of the engineering school of the University of California, Berkeley. Donald Knuth discusses the theorem in the third volume of *The Art of Computer Programming* in connection with a method of sorting called "shellsort." In my *The Last Recreations*, Chapter 11, I describe a clever card trick based on the theorem.

Is it possible to arrange a deck so that if you spell the name of each card by moving a card from top to bottom for each letter, then turning over the card at the end of the spell and discarding it, it will always be the card you spelled? For example, can you so arrange the cards that you can first spell all the spades, taking them in order from ace through king, then do the same thing with the hearts, clubs, and diamonds? You might imagine it would take a long time to find out how to arrange the deck, assuming it is possible to do so, in a way that permits the spelling of all 52 cards. Actually, finding the order is absurdly easy. First arrange the deck from top down in the order that is the reverse of your spelling sequence. Take the King of Diamonds from the top of the deck, then take the queen, place it on top of the king, and spell "Queen of Diamonds" by moving a card at each letter from bottom to top. In brief, you are reversing the spelling procedure. Continue in this way until the new deck is formed. You are now all set to spell every card in the predetermined order. Of course you can do the same thing with smaller packets, such as the thirteen spades, or with cards bearing pictures, say of animals whose names you spell.

Remember the old brainteaser about two glasses, one filled with water, the other with wine? You take a drop of water, put it into the wine, stir, then take a drop of the mixture, move it back to the water, and stir. Is there now more or less wine in the water than water in the wine? The answer is that the two quantities are exactly equal. The simplest proof is to realize that, after the transfers, the amounts of liquid in each glass remain the same. So the quantity missing from the water is replaced by wine, and amount of wine missing from the other glass is replaced by the same amount of water.

This is easily modeled with cards. Divide the deck into two halves, one of all red cards, the other of all black. Randomly remover n red cards, insert them anywhere in the black half, and shuffle. Now randomly remove n cards from the half you just shuffled, put them back among the reds, and shuffle. Inspection will show that the number of black cards in the red half exactly equals the number of red cards in the black half. It doesn't matter in the least if the red and black portions are not equal at the start.

Closely related to this demonstration is the following trick. Cut a deck exactly in half, turn over either half and shuffle the two parts together. Cut the mixed-up deck in half again, and turn over either half. You'll find that the number of face-down cards in either half exactly equals the number of face-down cards in the other half. The same is true, of course, for the face-up cards. Do you see why this is the case? The trick is baffling to spectators if they don't know that the deck is initially divided exactly in half, and if you secretly turn over one half as you spread its cards on the table.

Playing cards provide a wealth of counterintuitive probability questions. The notorious Monty Hall problem can be modeled with cards. There are three cards face down on the table and you are told that one card only is an ace. Put a finger on a card. Clearly the chance you have selected the ace is 1/3. A friend now secretly peeks at all three cards and turns face up a card that is not the ace. Two cards remain face down, one of which you know is the ace. What now is the probability your finger is on the ace? Many persons think the probability has risen from $1/3$ to $1/2$. A little reflection should convince you that it remains $1/3$ because your friend can always turn a non-ace. Now switch your finger from the card it is on to the other card. The probability you have now chosen the ace jumps form $1/3$ to $2/3$. This is obvious from the fact that the card you first selected has the probability of $1/3$ being the ace. Because the ace must be one of the two face-down cards, the two probabilities must add to 1 or certainty.

A similar seeming paradox also involves three face-down cards dealt from a shuffled deck. A friend looks at their faces and turns over two that are the same color. What's the probability that the remaining face-down card is the same color as the two face-up cards? You might think it is $1/2$. Actually it is $1/4$. Here's the proof. The probability that three randomly selected cards are the same color is two out of eight equal possibilities, or $1/4$. Subtract $1/4$ from 1 (the card must be red or black) and you get $3/4$ for the probability that the face-down card differs in color from the two face-up cards. This is the basis for an ancient sucker bet. If you are the operator, you can offer even odds that the card is of opposite color from the two face-up cards, and win the bet three out of four times.

Here's a neat problem involving a parity check. Take three red cards from the deck. Push one of them back into the pack and take out three black cards. Push one of them back into the deck and remove three reds. Continue in this manner. At each step you randomly select a card of either color, return it to the deck and remove three cards of opposite color. Continue as long as you like. When you decide to stop you will be holding a mixture of reds and blacks. Is it possible that the number of black cards you hold will equal the number of reds? Unless you think of a parity check it might take a while to prove that the answer is no. After each step you will always have in your hand an odd number of cards, therefore the two colors can never be equal.

Magicians have discovered the following curiosity. Place cards with values ace through nine face down in a row in counting order, ace at the left. Remove a card from either end of the row. Take another card from either end. Finally, take a third card from either end. Add the values of the three cards, then divide by six to obtain a random number n. Count the cards in the row from left to right, and turn over the nth card. It will always be the four!

I leave it to readers to figure out why this works and perhaps to generalize it to longer rows of numbers. For example, use twelve cards with values $2, 3, 4, 5, 6, 7, 8, 9, 10, \mathrm{J}, \mathrm{Q}, \mathrm{K}$ to make the row. Take a card three times from either end, divide their sum by 9, and call the result n. The nth card from the left will always be the five.

A classic card task, going back more than two centuries, is to arrange all the aces, kings, queens, and jacks—sixteen cards in all—in a square array so that no two cards of the same value, as well as no two cards of the same suit, are in the same row, column, or diagonal. Counting the number of different solutions is not trivial. W. W. Rouse Ball, in his classic *Mathematical Recreations and Essays*, said there are 72 fundamental solutions, not counting rotations and reflections. This is a mistake that persisted through the book's eleventh edition, but was dropped from later editions revised by H. S. M. Coxeter. Dame Kathleen Ollerenshaw, a noted British mathematician who was once Lord Mayor of Manchester, found there are twice as many fundamental solutions, 144, making the number of solutions including rotations and reflections $8 \times 144 = 1{,}152$. She recently described a simple procedure for generating all $1,152$ patterns in an article written for the blind. (Dame

Ollerenshaw, now 87, is slowly losing her vision, and energetically learning how to read Braille.)

This is her procedure. Number the sixteen positions of the square from 1 through 16, left to right, top down. Place an arbitrary card, say the Ace of Spades, in position 1, the top left corner. A second ace, say the Ace of Hearts, goes in the second row. It can't go in the same column or diagonal as the Ace of Spades, so it must go in either space 7 or 8. Place it arbitrarily in space 7. Two aces remain to go in rows 3 and 4. Put the Ace of Diamonds in the third row. It can go only in space 12. The Ace of Clubs is now forced into space 14 of the bottom row. Had the second ace gone in space 8, the last two aces would be forced into spaces 10 and 15.

Consider the other three spades. They can't go in the top row or leftmost column, or in a main diagonal. This forces them into spaces 4, 10, and 15. Arbitrarily place the King of Spades in 4, the Queen of Spades in 10, and the Jack of Spades in 15. The pattern now looks like this:

A♠			
		A♥	K♠
	Q♠		A♦
	A♣	J♠	

The remaining nine cards are forced into spaces that complete the following pattern:

A♠	K♥	Q♦	J♣
Q♣	J♦	A♥	K♠
J♥	Q♠	K♣	A♦
K♦	A♣	J♠	Q♥

Multiply the number of choices at each step, $16 \times 3 \times 2 \times 2 \times 3 \times 2$, and you get the total of 1,152 patterns.

For more examples of mathematical theorems, problems, and tricks with playing cards, see my Dover paperback *Mathematics, Magic, and Mystery*, and Karl Fulves's *Self-Working Card Tricks*, also a Dover soft-cover, and the following chapters in my collections of *Scientific American* columns: *The Scientific American*

Book of Mathematical Puzzles and Diversions, Chapter 10; *Mathematical Carnival*, Chapter 10 and 15; *Mathematical Magic Show*, Chapter 7; *Wheels, Life, and Other Mathematical Amusements*, Chapter 19; *Penrose Tiles to Trapdoor Ciphers*, Chapter 19, and *The Last Recreations*, Chapter 2.

Now for two puzzles that can be modeled with cards. Solutions will appear in the next issue.

1. Arrange nine cards as shown in Figure 1. Assume the aces have a value of 1. Each row, each column, and one diagonal has a sum of 6. The task is to alter the positions of three cards so that the matrix is fully magic for all rows, columns, and diagonals.

A	2	3
3	A	2
2	3	A

Figure 1

2. Nine cards arranged and shown in Figure 2 have the property of minimizing the sum of all absolute differences between each pair of cells that are adjacent vertically and horizontally. Assume that the matrix is toroidal; that is, it wraps around in both directions. The sum of the differences is 48. This was proved minimal by Friend Kirstead, Jr., in the *Journal of Recreational Mathematics* (18, 1985–86, 301). The challenge is to take nine cards of distinct values (court cards may be used) and form a toroidal square that will *maximize* the sum of all absolute differences.

A	2	3
4	5	6
7	8	9

Figure 2