all roots of f_0. By the lemma, the expression

$$\int h \operatorname{sgn} f_0 = \sum_{i=1}^{n} \sigma_i \int_{x_{i-1}}^{x_i} h = \sum_{i=1}^{n} \sigma_i \phi_i(h) \qquad (\sigma_i = \pm 1)$$

vanishes for each $h \in P$, because otherwise one could secure the inequality $\int |f_0 - \lambda h| < \int |f_0|$ with an appropriate choice of $h \in P$ and λ. If $\{g_1, \cdots, g_n\}$ is a basis for P then the matrix $[\phi_i(g_j)]$ is singular, because $\sum_i \sigma_i \phi_i(g_j) = 0$. Thus we can find a nonzero n-tuple $(c_1, \cdots, c_n)$ such that $\sum_j c_j \phi_i(g_j) = 0$. This equation implies that the nonzero function $h = \sum_j c_j g_j$ has the property

$$\int_{x_{i-1}}^{x_i} h = \phi_i(h) = \phi_i\left(\sum_j c_j g_j\right) = \sum_j c_j \phi_i(g_j) = 0.$$

But then h must possess at least one root in each interval (x_{i-1}, x_i), in contradiction with the Haar property.

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LIMIT OF A FUNCTION AND A CARD TRICK

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This article intends to explain some mathematical ideas which cause a card trick to work. We believe there is some pedagogical value in this note in regard to the concept of limit.

Choose 21 cards and put them down, face up, one at a time, in three sets. We shall explain the idea more thoroughly. Put down three cards first and add one card to each. Continue in this manner until three sets of equal number of cards are obtained. While doing this we ask a friend to choose a card and to tell us only in which set the chosen card is. Then we put the set, in which the chosen card is, between the other two sets. We repeat the same thing and ask our friend again to tell us in which set the chosen card is. Again we put this set between the other two sets. The third time we sort the cards into three sets as before and we observe that the chosen card is exactly at the middle of the set in which the chosen card has gone.

The reader may have already seen this trick or knows it. We are not trying to teach a card trick. We would like to study the mathematical explanation of the trick and look into some generalizations of it.

Let us say that the position of the chosen card is a function of the number of times we have gone through the process of putting cards into three sets. In this particular case we observe that $8 \leqq f(1) \leqq 14$. This is quite obvious since the chosen card is in the set of seven cards which is in between the other two sets of seven cards.

Now by letting $f(1)=14$ or $f(1)=8$ we shall find that $10 \leqq f(2) \leqq 12$. Indeed this is easily done for a set of 21 cards. We shall obtain a technique of finding the bounds of $f(k)$ for the general case.

Next we see that $11 \leqq f(3) \leqq 11$. Thus $f(3)=11$. We may say

$$\lim_{x\to 3} f(x) = 11.$$

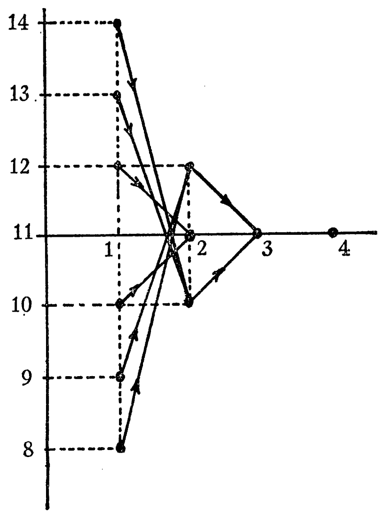

FIG. 1.

We shall draw a diagram giving the bounds of $f(x)$ for each x (Fig. 1). We should really be less sloppy with notations. For each x the set $\{f(x)\}$ is a bounded set, both above and below. Thus we should use the ideas of least upper bounds and greatest lower bounds which, in fact, belong to the set $\{f(x)\}$ in this problem. The diagram shows the least upper bound and greatest lower bound of $\{f(x)\}$ for $x=1$, 2, and 3. We also use an arrow in the diagram to show:

For $f(1)=14$ or $f(1)=13$ we get $f(2)=10$. But if $f(1)=12$ then $f(2)=11$.

For $f(1)=8$ or $f(1)=9$ we get $f(2)=12$. But if $f(1)=10$, then $f(2)=11$.

Thus there are several ways that $f(x)$ approaches its limit.

Now let us look into an example where the limit does not exist. Chose 24 cards and do the same thing with them, i.e., set them into three sets and choose a card. Then we put the set in which the chosen card appears, between the other two. We repeat as many times as we would like. We observe that

$$9 \leqq f(1) \leqq 16,$$
$$11 \leqq f(2) \leqq 14,$$
$$12 \leqq f(3) \leqq 13,$$
$$12 \leqq f(4) \leqq 13,$$
$$\cdots\cdots\cdot.$$

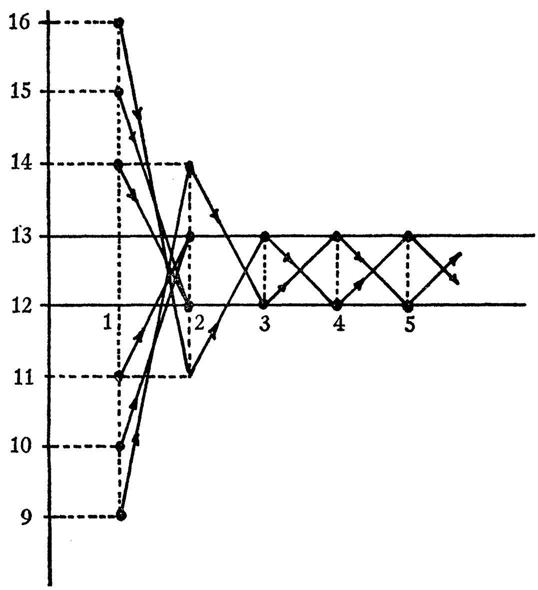

FIG. 2.

Thus the least upper bound and greatest lower bound of any set $\{f(x)\}$ never become the same. Here again we supply a diagram (Fig. 2). The diagram illustrates the following observation:

For $f(1)=16$ we have $f(2)=11$. But for $f(1)=15$ or $f(1)=14$ we have $f(2)=12$.

For $f(1)=9$ we have $f(2)=14$. But for $f(1)=10$ or $f(1)=11$ we have $f(2)=13$. Indeed we can also do a card trick in this case. After setting the cards down three times, we know that the chosen card will be in the position 12 or 13. In this way we keep the two possible cards in mind. In the next sorting of the cards we can choose the correct one.

Now let us look into a more general case. In what follows every small letter denotes a positive integer. Let the number of cards be $3q$. Then we look for a number k such that $\lim_{x\to k} f(x)$ exists, where x is defined as before. Here k indicates the number of times which is necessary and perhaps sufficient to put the cards into three sets in order to get the chosen card in the middle. It is clear that $q+1 \leqq f(1) \leqq 2q$.

Consider the case $f(1)=2q$. We have to divide the cards into three sets as was described. We know that $2q=3c+r$, where $0\leqq r<3$. But we shall write instead

$$(1)\qquad 2q = 3c_2 - r_2, \qquad 0 \leqq r_2 < 3.$$

The reader will discover the advantage of this choice as we proceed. We observe that (1) implies that

$$(2)\qquad q - r_2 = 3m_2.$$

The equalities (1) and (2) show that first we put down three sets of c_2 cards and three sets of m_2 cards on the top of them (Fig. 3). Each set is indicated by a rectangle and the number of cards are written inside the rectangle. The equality (1) shows that the chosen card is on the top of one of the sets of c_2 cards. Thus in this case

$$(3)\qquad f(2) = q + 1 + m_2.$$

Now let $f(1)=q+1$. Then by (2) we have

$$q + 1 = 3m_2 + r_2 + 1, \qquad 0 \leqq r_2 < 3.$$

In this case we first put down three sets of m_2 cards each (Fig. 4). Then we observe that the rest of the cards, i.e.,

$$3q - 3m_2 = 3q - (q - r_2) = 2q + r_2$$

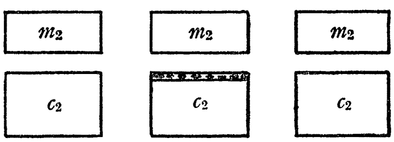

FIG. 3.

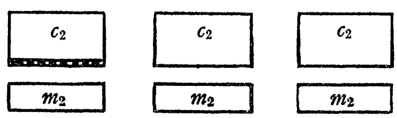

FIG. 4.

is divisible by 3; in fact $2q+r_2=3c_2$.

Thus a set of c_2 cards is put over each set of m_2 cards. The chosen card is at the bottom of one of these sets. Thus $f(2)=q+c_2$. But (1) and (2) imply that $3q=3c_2+3m_2$ or $q=c_2+m_2$. Therefore

$$(4)\qquad f(2) = 2q - m_2.$$

The values of $f(2)$ in (3) and (4) are, respectively, the greatest lower bound and the least upper bound of the set $\{f(2)\}$. Thus $q+1+m_2\leqq f(2)\leqq 2q-m_2$, where

$$(5)\qquad q = 3m_2 + r_2, \qquad 0 \leqq r_2 \leqq 3.$$

The reader may easily show that $q+1+m_2 \leqq 2q-m_2$. The equality holds if and only if $q=1$.

Now suppose that in a similar way as to what was described so far we obtain

$$q + 1 + m_{k-1} \leqq f(k-1) \leqq 2q - m_{k-1},$$

where $q+m_{k-2}=3m_{k-1}+r_{k-1}$, $0 \leqq r_{k-1} < 3$ and

$$q + 1 + m_{k-1} < 2q - m_{k-1}.$$

Let us for simplicity write $a \leqq f(k-1) \leqq b$. As before let $f(k-1)=b$. Then $b=3c_k-r_k$ where $0 \leqq r_k < 3$. This implies that

$$3q - b - r_k = 3m_k. \tag{6}$$

Again we observe that putting the cards into three sets has two stages. First three sets of c_k cards are put down (Fig. 5). Then three sets of m_k cards are put over them. As before we observe that the chosen card is at the top of one of the sets of c_k cards. This implies that the greatest lower bound of $\{f(k)\}$ is $q+1+m_k$.

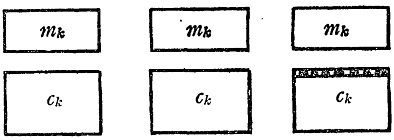

FIG. 5.

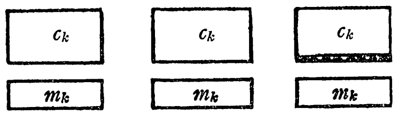

FIG. 6.

Now let $f(k-1)=a$. Then we observe that $a+b=3q+1$ or $a=3q-b+1$. This and (6) imply that $3q-b=3m_k+r_k$. Therefore $a-1=3m_k+r_k$ where $0 \leqq r_k < 3$. Here again we put down three sets of m_k cards and a set of c_k cards on top of each of them (Fig. 6). The chosen card is at the bottom of one of the sets which has c_k cards. Thus $f(k)=q+c_k$.

Since $q=c_k+m_k$, it follows that $f(k)=2q-m_k$ and this is the least upper bound of $\{f(k)\}$. Thus we get $q+1+m_k \leqq f(k) \leqq 2q-m_k$ where

$$q + m_{k-1} = 3m_k + r_k, \qquad 0 \leqq r_k < 3.$$

This gives a technique of getting $f(k)$ from $f(k-1)$. The reader may show that $m_k \geqq m_{k-1}$ and $2q-m_k \geqq q+1+m_k$. Let us study the case that $q=2h+1$. Here the inequality $q+1+m_k \leqq 2q-m_k$ implies that $m_k \leqq h$. The equality holds if and only if $m_k=h$. But, in general, we have $m_{k+1} \geqq m_k$. Thus $q+1+m_k=2q-m_k$ if and only if $m_{k+1}=m_k=h$. Then this k is the number of times necessary and sufficient to put the cards into three sets in order to have $f(k)=q+h+1$.

In the case where $q=2h$ a necessary and sufficient condition for $q+2+m_k = 2q-m_k$ is that $m_k=m_{k-1}=h-1$. We omit the proof since it is very similar to the previous case.

It would be interesting if an explicit formula for k in terms of h could be obtained. We leave it as a problem.

The reader might consider the generalization to the cases of putting a set of cards into 5, 7, $\cdots$, $2h+1$ sets. Are there other generalizations?

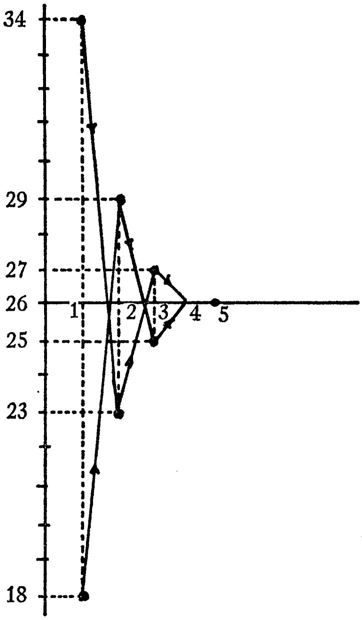

FIG. 7.

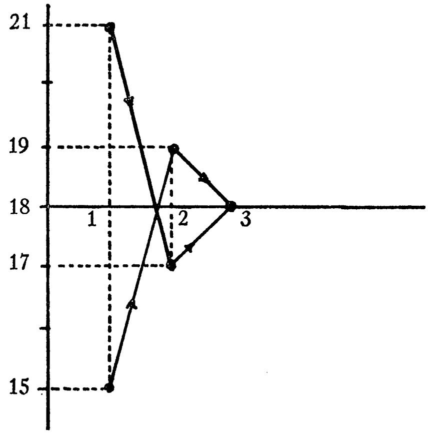

FIG. 8.

We conclude this article by giving diagrams for a set of 51 cards (Fig. 7) and putting them into 3 sets, and 35 cards and putting them into 5 sets (Fig. 8). We observe that k for 51 cards and 3 sets is 4. We also see that k for 35 cards and 5 sets is 2.

A FUNCTION WHOSE VALUES ARE INTEGERS

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Introduction. In this paper, using the determinants of certain triangular matrices, we prove the following

THEOREM. *For any positive integers k and n,*

$$((2k+1)!/3\cdot 2^k k!)\left(\sum_{r=1}^{n} r^{2k} \Big/ \sum_{r=1}^{n} r^2\right) \tag{1}$$

is an integer.

Proof. For any integers k and w, $k>0$, notice that

$$\text{(a)}\quad (w+1)^{2k+1} - w^{2k+1} = 1 + \sum_{r=1}^{2k} \binom{2k+1}{r} w^{2k+1-r}, \tag{2}$$