Card Tricks
Magic or Mathematics?

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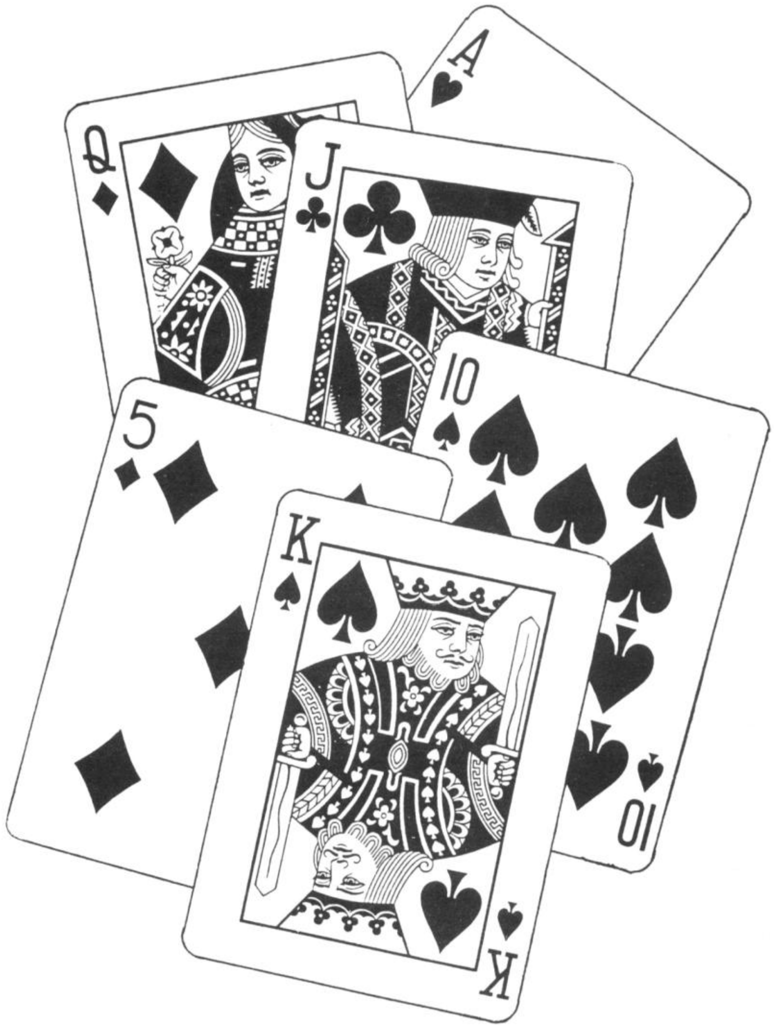
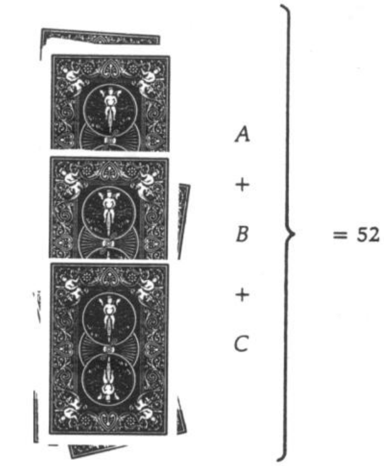


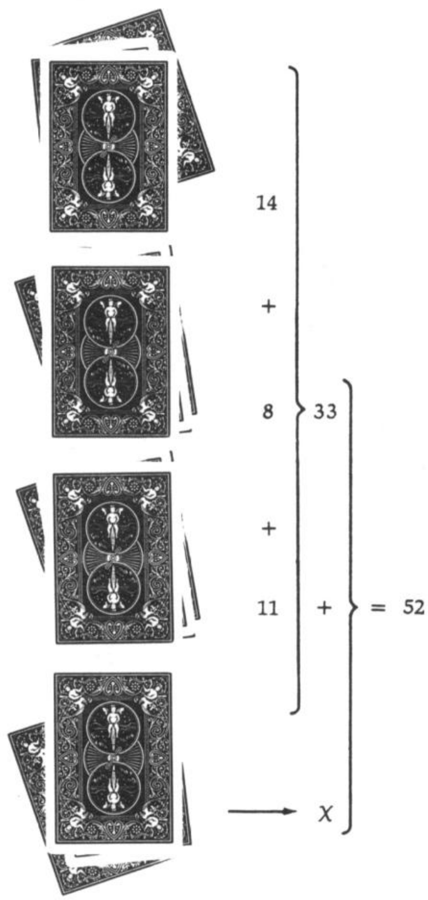
On the one hand, when viewed by students, magicians and mathematics have something in common – both have a "bag of tricks" for their performances. On the other hand, they differ drastically – magicians do not open up their "bag of tricks" and show the audience how their tricks work, whereas mathematicians have nothing to hide and they are very happy to show and prove their "tricks" by means of mathematics. Favorite among the magician's "bag of tricks," card tricks can be derived from basic properties of beginning algebra. In the examples and exercises that follow, you will see how mathematics can be used to develop and prove the techniques of some card tricks.



In all of these card tricks, there are two Basic Rules that are used to develop the equations for solving these card tricks. The following is the first rule:

Rule #1: In forming several piles of cards from a deck of cards, the sum of the number of cards in these piles is the same as the number of cards in the deck.

Example #1: If three piles of cards are formed with 14, 8, and 11 cards, respectively, from a standard deck of 52 cards, how many cards are left over?



Solution: Let X = the number of cards left over; these cards can then be used to form a fourth pile of cards, so that by **Rule #1,**

$$14 + 8 + 11 + X = 52,$$

$$33 + X = 52,$$

$$\text{or,} \quad X = 19.$$

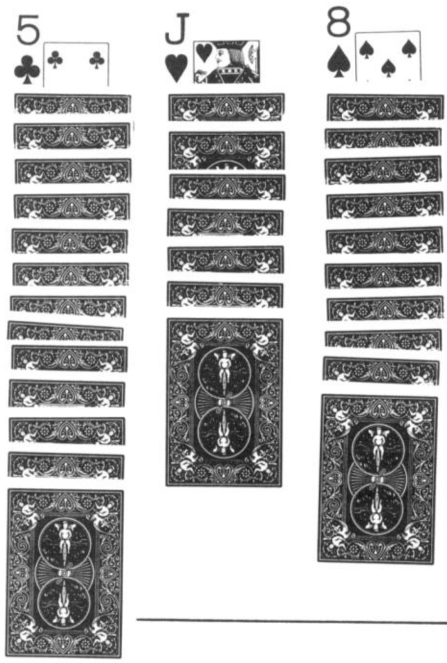
Hence, there are 19 cards left over.

Suppose that to form these three piles in **Example #1,** you use the following procedure: each ace counts as a "1," two through ten count as their face value, jacks count as "11," queens as "12," and kings as "13." In forming each pile, cards are dealt one-by-one and the cards are counted by starting the count with the value of the first card dealt and continuing until 18 is reached.

For the piles in **Example #1,** the first card dealt for the first pile had to be a 5

$$\underbrace{18,\ 17,\ 16,\ 15,\ 14,\ 13,\ 12,\ 11,\ 10,\ 9,\ 8,\ 7,\ 6,\ 5}_{14 \text{ cards}}$$

or $18 + 1 - 14$, the first card dealt for the second pile had to be a jack $(18 + 1 - 8)$, and the first card of the third pile had to be an 8 $(18 + 1 - 11)$. Are these numbers correct?



Similarly, if one knows the value of the first card dealt for each pile and that the count stops at 18, this is enough information to determine the number of cards in each pile WITHOUT actually counting the number of cards:

First pile: $(18 + 1) - 5 = 14$ cards,

Second pile: $(18 + 1) - 11 = 8$ cards,

Third pile: $(18 + 1) - 8 = 11$ cards,

Notice that in each pile,

(value of last card + 1) −(value of first card)=
number of cards in pile

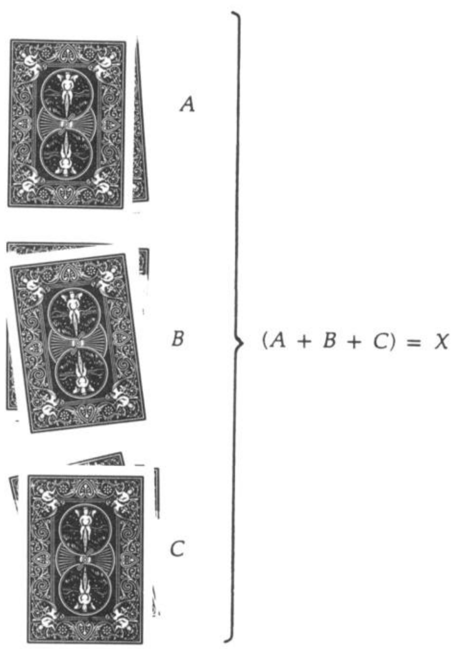
This is the second rule that is used to develop equations for solving these card tricks.

Rule #2: In forming a pile of cards, if F is the value of the first card dealt and L is the value of the last card dealt, then the number of cards N in the pile is

$$N = (L + 1) - F$$

Let's now use these two rules, along with some algebra, and develop the first card trick.

Example #2: (Card Trick #1)
Suppose that you have a deck of 42 cards and you form three piles of cards, counting the value of the cards as described above, until 13 is reached for each pile. As the cards are being dealt, they are placed face-down on the pile. Develop an equation that will tell you the sum of the values of the three bottom cards if you know the number of cards left over.



Solution: Let A, B, and C be the value of the bottom cards of the first, second, and third piles. By **Rule #2,**

$(13 + 1) - A$ is the number of cards in the first pile

$(13 + 1) - B$ is the number of cards in the second pile

$(13 + 1) - C$ is the number of cards in the third pile

If there are X cards left over, then by **Rule #1**

$$(14 - A) + (14 - B) + (14 - C) + X = 42,$$

$$(3)(14) - A - B - C + X = 42,$$

$$42 - (A + B + C) + X = 42,$$

$$X = (A + B + C).$$

This last equation tells us that X, the number of cards left over, is the same as the sum of the values of the three bottom cards of the three piles. This means that if someone sets up the three piles and you do NOT see the person doing it, you can tell that person the sum of the values of the three bottom cards by counting out the number of cards remaining. (To convince yourself that this card trick works, try it a few times; then try it on your friends!)

You Try It #1. (Card Trick #2) As in **Example #2,** form three piles of cards from a deck of 42 cards, where the cards have the following values: each ace counts as "1," twos through tens count as their face value, and each jack, queen, and king counts as "10." Develop an equation that will tell you the sum of the values of the bottom cards in terms of the number of cards left over.

You Try It #2. (Card Trick #3) You have a deck of 39 cards and you form three piles of cards (assigning values to the cards as you did in **You Try It #1** above) by the following methods:

First Pile: Count out cards until you reach "10."

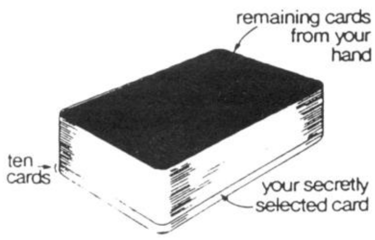
Second Pile: Count out cards until you reach "12."

Third Pile: Count out cards until you reach "14."

Develop an equation that will tell you the value of the bottom card of one of the piles if you are told the sum of the values of the bottom cards of the other two piles and how many cards are left over.

You Try It #3. (Card Trick #4) For this card trick, two people, you and the "dealer" (each stationed in different rooms), are needed. This trick uses a standard deck of 52 cards and the cards have the following values: an ace counts as "1," two through ten count as their face value , a jack counts as "11," a queen counts as "12," and a king counts as "13." Again, piles of cards are formed by the dealer (out of your sight) in the following manner: the top card of the deck is dealt face-up, its value is noted and additional cards are dealt one by one (also face-up) and counted until "13" is reached. The process is continued until three piles are formed and the undealt cards are retained by the dealer. The dealer then turns all of the piles face-down and then turns over the top card of two of the piles.

You then return to the scene and take the remaining cards from the dealer (all of the cards that are not in the three piles) and deal out a number of cards equal to 10 more than the sum of the values of the two exposed cards. The number of cards you have left is the numerical value of the unexposed top card of the third pile. Develop an equation that explains why this trick works.



Stack of cards face down

You Try It #4: (Card Trick #5) Suppose that you have a standard deck of 52 cards and the cards are assigned the same values as in **You Try It #3 (Card Trick #4).** Start building a pile of cards by placing cards face-up. At the same time, claim that you are memorizing this sequence of cards, and challenge others to also memorize the sequence of cards! Even though you are not actually memorizing this sequence of cards, you secretly pick out and remember one particular card, hereby known as "your card." Then deal out EXACTLY ten more cards face-up on top of "your card." Turn over this face-up pile of cards and place that pile UNDER the stack of cards in your hand. Now form three piles of cards where each pile is formed (by dealing from the top of the deck) as follows: the value of the top card of the deck is noted, the card is placed face-down, and additional cards are dealt one by one (also face-down) and counted until "13" is reached. Then turn over these three piles and determine the sum, S, of the values of the three exposed cards. Now ask someone who was memorizing the sequence of the cards, if they know what the card is that is in position S from the top of the remaining cards in your hand. The card that is in position S from the top of the remaining cards in your hand is "your card." Why does that sum, S, tell you the position of "your card" in the remainder of the deck in your hand?

You Try It #5: In the five card tricks presented above, there are many "variable items": the number of cards in a deck, the values assigned to the cards, the number of piles formed, and the stopping point when forming a pile of cards. In each of the five card tricks, make changes in some (or all) of these "variable items" and develop other card tricks and appropriate equations for explaining the techniques used in presenting the card tricks.

With all of these card tricks, now you can play two roles: as a magician you have some card tricks to show to your friends, and as a mathematician you have mathematics to verify the techniques used in performing these tricks. GOOD LUCK!

References

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Answers

Answers to "You Try It"

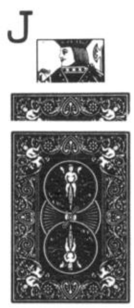
1. Using A, B, C, and X as in **Example #2,** then

$$(14 - A) + (14 - B) + (14 - C) + X = 42$$

which is the same as the first equation of the solution of **Card Trick #1.** Hence,

$$X = (A + B + C),$$

so that the sum of the values of the three bottom cards is the same as the number of cards remaining.



2. Use A, B, and C to represent the values of the bottom cards respectively of the first, second, and third piles, and X to represent the number of cards remaining. By **Rules #1** and **#2,** then

$$(11 - A) + (13 - B) + (15 - C) + X = 39,$$

$$(11 + 13 + 15) - A - B - C + X = 39,$$

$$39 - (A + B + C) + X = 39,$$

$$X = (A + B + C).$$

Suppose that you are told the value of the bottom cards of the first and third piles, A and C. Then, by solving the last equation for B,

$$B = X - (A + C).$$

Hence, to find the value of the bottom card of the second pile, namely B, subtract the sum of the values of the bottom cards of the other two piles from X, the number of cards remaining.

3. Let A, B, and C be the values of the top cards of the first, second, and third piles respectively, so that the number of cards in these piles is $(14 - A)$, $(14 - B)$, and $(14 - C)$. If X is the number of cards remaining, then

$$(14 - A) + (14 - B) + (14 - C) + X = 52,$$

$$42 + X - (A + B + C) = 52,$$

$$X - (A + B + C) = 10.$$

Suppose that you know the value of the top cards of the first and the second piles, so that you are able to determine (A + B). Then, by solving the last equation for C you obtain

$$C = X - (10 + A + B).$$

Hence, the value of the unexposed top card of the third pile, C, is obtained by subtracting 10 more than the sum of the values of the two exposed cards, $10 + A + B$, from the number of remaining cards, X. In other words, if you deal out (take away) from the remaining cards, X, a number of cards equal to 10 more than the sum of the values of the two exposed cards, $10 + A + B$, the number of cards that you have left, $X - (10 + A + B)$, is the value of the unexposed top card of the third pile, C.

4. After you have determined "your card" and prior to forming the three piles, the deck of cards consists of two parts: the bottom part of the deck which has the 10 cards that you counted out and the top part of the deck which consists of the other 42 cards, the 42nd card from the top being "your card." Let A, B, and C represent the value of the bottom card of each of the piles where the cards are dealt face-up; when these three piles are turned over, then the A, B, and C represent the value of the three exposed cards, which means that $S = A + B + C$. Let X represent the number of cards left over from the top part of the deck after the three piles have been formed, so that when accounting for all of the cards of the deck,

$$\begin{aligned} 10 + X + (14 - A) + (14 - B) + (14 - C) &= 52, \\ 10 + X + 42 - (A + B + C) &= 52, \\ 52 + X - S &= 52, \\ X - S &= 0, \\ X &= S. \end{aligned}$$

The last equation tells you that X, the number of cards left over from the top part of the deck after the three piles have been formed, is the same as S, the number of cards you have to count down to in order to find "your card." Remember that the bottom card of those X cards is "your card" so you must count out S cards to get to it.