

This column is a place for those bits of contagious mathematics that travel from person to person in the community, because they are so elegant, surprising, or appealing that one has an urge to pass them on.

Contributions are most welcome.

The Best Card Trick

Michael Kleber

You, my friend, are about to witness the best card trick there is. Here, take this ordinary deck of cards, and draw a hand of five cards from it. Choose them deliberately or randomly, whichever you prefer—but do not show them to me! Show them instead to my lovely assistant, who will now give me four of them: the $7\spadesuit$, then the $Q\heartsuit$, the $8\clubsuit$, the $3\diamondsuit$. There is one card left in your hand, known only to you and my assistant. And the hidden card, my friend, is the $K\spadesuit$.

Surely this is impossible. My lovely assistant passed me four cards, which means there are 48 cards left that could be the hidden one. I received the four cards in some specific order, and by varying that order my assistant could pass me some information: one of $4! = 24$ messages. It seems the bandwidth is off by a factor of two. Maybe we are passing one extra bit of information illicitly? No, I assure you: the only information I have is a sequence of four of the cards you chose, and I can name the fifth one.

The Story

If you haven't seen this trick before, the effect really is remarkable; reading it in print does not do it justice. (I am forever indebted to a graduate student in one audience who blurted out "No way!" just before I named the hidden card.) Please take a moment to ponder how the trick could work, while I relate some history and delay giving away the answer for a page or two. Fully appreciating the trick will involve

a little information theory and applications of the Birkhoff-von Neumann theorem as well as Hall's Marriage theorem. One caveat, though: fully appreciating this article involves taking its title as a bit of showmanship, perhaps a personal opinion, but certainly not a pronouncement of fact!

The trick appeared in print in Wallace Lee's book *Math Miracles*,¹ in which he credits its invention to William Fitch Cheney, Jr., a.k.a. "Fitch." Fitch was born in San Francisco in 1894, son of a professor of medicine at Cooper Medical College, which later became the Stanford Medical School. After receiving his B.A. and M.A. from the University of California in 1916 and 1917, Fitch spent eight years working for the First National Bank of San Francisco and then as statistician for the Bank of Italy. In 1927 he earned the first math Ph.D. ever awarded by MIT; it was supervised by C.L.E. Moore and titled "Infinitesimal deformation of surfaces in Riemannian space." Fitch was an instructor and assistant professor then at the University of Hartford (Hillyer College before 1957) until his retirement in 1971; he remained an adjunct until his death in 1974.

For a look at his extra-mathematical activities, I am indebted to his son Bill Cheney, who writes:

My father, William Fitch Cheney, Jr., stage-name "Fitch the Magician," first became interested in the art of magic when attending vaudeville shows with his parents in San Francisco in the early 1900s. He devoted countless hours to learning sleight-of-hand skills and other "pocket magic" effects with which to entertain friends and family. From the time of his initial teaching assignments at Tufts College in the 1920s, he enjoyed introducing magic effects into the classroom, both to il-

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¹Published by Seeman Printery, Durham, N.C., 1950; Wallace Lee's Magic Studio, Durham, N.C., 1960; Mickey Hades International, Calgary, 1976.

lustrate points and to assure his students' attentiveness. He also trained himself to be ambidextrous (although naturally left-handed), and amazed his classes with his ability to write equations simultaneously with both hands, meeting in the center at the "equals" sign.

Each month the magazine M-U-M, official publication of the Society of American Magicians, includes a section of new effects created by society members, and "Fitch Cheney" was a regular by-line. A number of his contributions have a mathematical feel. His series of seven "Mental Dice Effects" (beginning Dec. 1963) will appeal to anyone who thinks it important to remember whether the numbers 1, 2, 3 are oriented clockwise or counterclockwise about their common vertex on a standard die. "Card Sense" (Oct. 1961) encodes the rank of a card (possibly a joker) using the fourteen equivalence classes of permutations of $abcd$ which remain distinct if you declare $ac = ca$ and $bd = db$ as substrings: the card is placed on a piece of paper whose four edges are folded over (by the magician) to cover it, and examining the creases gives precisely that much information about the order in which they were folded.²

While Fitch was a mathematician, the five-card trick was passed down via Wallace Lee's book and the magic community (I don't know whether it appeared earlier in M-U-M or not.) The trick seems to be making the rounds of the current math community and beyond, thanks to mathematician and magician Art Benjamin, who ran across a copy of Lee's book at a magic show, then taught the trick at the Hampshire College Summer Studies in Mathematics program³ in 1986. Since then it has turned up regularly in "brain teaser" puzzle-friendly fo-

rums; on the *rec.puzzles* newsgroup, I once heard that it was posed to a candidate at a job interview. It made a recent appearance in print in the "Problem Corner" section of the January 2001 *Emissary*, the newsletter of the Mathematical Sciences Research Institute. As a result of writing this column, I am learning about a slew of papers in preparation that discuss it as well. It is a card trick whose time has come.

The Workings

Now to business. Our "proof" of impossibility ignored the other choice my lovely assistant gets to make: which of the five cards remains hidden. We can put that choice to good use. With five cards in your hand, there are certainly two of the same suit; we adopt the strategy that the first card my assistant shows me is of the same suit as the card that stays hidden. Once I see the first card, there are only twelve choices for the hidden card. But a bit more cleverness is required: by permuting the three remaining cards my assistant can send me one of only $3! = 6$ messages, and again we are one bit short.

The remaining choice my assistant makes is which card from the same-suit pair is displayed and which is hidden. Consider the ranks of these cards to be two of the numbers from 1 to 13, arranged in a circle. It is always possible to add a number between 1 and 6 to one card (modulo 13) and obtain the other; this amounts to going around the circle "the short way." In summary, my assistant can show me one card and transmit a number from 1 to 6; I increment the rank of the card by the number, and leave the suit unchanged, to identify the hidden card.

It remains only for me and my assistant to pick a convention for representing the numbers from 1 to 6. First, totally order a deck of cards: say ini-

tially by rank, $A23 \dots JQK$, and break ties by ordering the suits as in bridge (*i.e.*, alphabetical) order, $\clubsuit \diamond \heartsuit \spadesuit$. Then the three cards can be thought of as smallest, middle, and largest, and the six permutations can be ordered, *e.g.*, lexicographically.⁴

Now go out and amaze (and illuminate⁵) your friends. But, please: just make sure that you and your assistant agree on conventions and can name the hidden card flawlessly, say 20 times in a row, before you try this in public. As we saw above, it's not hard to name the hidden card half the time—and it's tough to win back your audience if you happen to get the first one wrong. (I speak, sadly, from experience.)

The Big Time

Our scheme works beautifully with a standard deck, almost as if four suits of thirteen cards each were chosen just for this reason. While this satisfied Wallace Lee, we would like to know more. Can we do this with a larger deck of cards? And if we replace the hand size of five with n , what happens?

First we need a better analysis of the information-passing. My assistant is sending me a message consisting of an ordered set of four cards; there are $52 \times 51 \times 50 \times 49$ such messages. Since I see four of your cards and name the fifth, the information I ultimately extract is an unordered set of five cards, of which there are $\binom{52}{5}$, which for comparison we should write as $52 \times 51 \times 50 \times 49 \times 48/5!$. So there is plenty of extra space: the set of messages is $\frac{120}{48} = 2.5$ times as large as the set of situations. Indeed, we can see some of that slop space in our algorithm: some hands are encoded by more than one message (any hand with more than two cards of the same suit), and some messages never get used (any message which contains the card it encodes).

²This sort of "Purloined Letter" style hiding of information in plain sight is a cornerstone of magic. From that point of view, the "real" version of the five-card trick secretly communicates the missing bit of information; Persi Diaconis tells me there was a discussion of ways to do this in the late 1950s. For our purposes we'll ignore these clever but non-mathematical ruses.

³Unpaid advertisement: for more information on this outstanding, intense, and enlightening introduction to mathematical thinking for talented high-school students, contact David Kelly, Natural Science Department, Hampshire College, Amherst, MA 01002, or dkelly@hampshire.edu.

⁴For some reason I personally find it easier to encode and decode by scanning for the position of a given card: place the smallest card in the left/middle/right position to encode 12/34/56, respectively, placing medium before or after large to indicate the first or second number in each pair. The resulting order $sml, slm, msl, lsm, mls, lms$ is just the lex order on the inverse of the permutation.

⁵If your goal is to confound instead, it is too transparent always to put the suit-indicating card first. Fitch recommended placing it $(i \bmod 4)$ th for the i th performance to the same audience.

Generalize now to a deck with d cards, from which you draw a hand of n . Calculating as above, there are $d(d-1)\cdots(d-n+2)$ possible messages, and $\binom{d}{n}$ possible hands. The trick really is impossible (without subterfuge) if there are more hands than messages, *i.e.*, unless $d \leq n! + n - 1$.

The remarkable theorem is that this upper bound on d is always attainable. While we calculated that there are enough messages to encode all the hands, it is far from obvious that we can match them up so each hand is encoded by a message using only the n cards available! But we can; the $n = 5$ trick, which we can do with 52 cards, can be done with a deck of 124. I will give an algorithm in a moment, but first an interesting nonconstructive proof.

The Birkhoff–von Neumann theorem states that the convex hull of the permutation matrices is precisely the set of *doubly stochastic matrices*: matrices with entries in $[0,1]$ with each row and column summing to 1. We will use the equivalent discrete statement that any matrix of nonnegative integers with constant row and column sums can be written as a sum of permutation matrices.⁶ To prove this by induction (on the constant sum) one need only show that any such matrix is entrywise greater than some permutation matrix. This is an application of Hall's Marriage theorem, which states that it is possible to arrange suitable marriages between n men and n women as long as any collection of k women can concoct a list of at least k men that someone among them considers an eligible bachelor. Applying this to our nonnegative integer matrix, we can marry a row to a column only if their common entry is nonzero. The constant row and column sums ensure that any k rows have at least k columns they consider eligible.

Now consider the (very large) $0-1$ matrix with rows indexed by the $\binom{d}{n}$ hands, columns indexed by the $d!(d-n+1)!$ messages, and entries equal to 1 indicating that the cards used in the message all appear in the hand. When we take d to be our upper

bound of $n! + n - 1$, this is a square matrix, and has exactly $n!$ 1's in each row and column. We conclude that some subset of these 1's forms a permutation matrix. But this is precisely a strategy for me and my lovely assistant—a bijection between hands and messages which can be used to represent them. Indeed, by the above paragraph, there is not just one strategy, but at least $n!$.

Perfection

Technically the above proof is constructive, in that the proof of Hall's Marriage theorem is itself a construction. But with $n = 5$ the above matrix has 225,150,024 rows and columns, so there is room for improvement. Moreover, we would like a *workable* strategy, one that we have a chance at performing without consulting a cheat sheet or scribbling on scrap paper. The perfect strategy below I learned from Elwyn Berlekamp, and I've been told that Stein Kulseth and Gadiel Seroussi came up with essentially the same one independently; likely others have done so too. Sadly, I have no information on whether Fitch Cheney thought about this generalization at all.

Suppose for simplicity of exposition that $n = 5$. Number the cards in the deck 0 through 123. Given a hand of five cards $c_0 < c_1 < c_2 < c_3 < c_4$, my assistant will choose c_i to remain hidden, where $i = c_0 + c_1 + c_2 + c_3 + c_4 \bmod 5$.

To see how this works, suppose the message consists of four cards which sum to $s \bmod 5$. Then the hidden card is congruent to $-s + i \bmod 5$ if it is c_i . This is precisely the same as saying that if we renumber the cards from 0 to 119 by deleting the four cards used in the message, the hidden card's new number is congruent to $-s \bmod 5$. Now it is clear that there are exactly 24 possibilities, and the permutation of the four displayed cards communicates a number p from 0 to 23, in "base factorial:" $p = d_1 1! + d_2 2! + d_3 3!$, where for lex order, $d_i \leq i$ counts how many cards to the right of the $(n-i)$ th are smaller than it.⁷ Decoding the hidden

card is straightforward: take $5p + (-s \bmod 5)$ and add 0, 1, 2, 3, or 4 to account for skipping the cards that appear in the message.⁸

Having performed the 124-card version, I can report that with only a little practice it flows quite nicely. Berlekamp mentions that he has also performed the trick with a deck of only 64 cards, where the audience also flips a coin: after seeing four cards the performer both names the fifth and states whether the coin came up heads or tails. Encoding and decoding work just as before, only now when we delete the four cards used to transmit the message, the deck has 60 cards left, not 120, and the extra bit encodes the flip of the coin. If the 52-card version becomes too well known, I may need to resort to this variant to stay ahead of the crowd.

And finally a combinatorial question to which I have no answer: how many strategies exist? We probably ought to count equivalence classes modulo renumbering the underlying deck of cards. Perhaps we should also ignore composing a strategy with arbitrary permutations of the message—so two strategies are equivalent if, on every hand, they always choose the same card to remain hidden. Calculating the permanent of the aforementioned 225,150,024-row matrix seems like a bad way to begin. Is there a good one?

Acknowledgments

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⁶Exercise: Do so for your favorite magic square.

⁷Or, my preference, d_i counts how many cards larger than the i th smallest appear to the left of it. Either way, the conversion feels perfectly natural after practicing a few times.

⁸Exercise: Verify that if your lovely assistant shows you the sequence of cards 37, 7, 94, 61, then the hidden card is the page number in this issue where the first six colorful algorithms converge.)