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The Gergonne p-pile problem and the dynamics of the function $x \mapsto \lfloor (x+r)/p \rfloor$

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Abstract

The Gergonne p-pile problem is concerned with repeatedly dealing (based upon some "fixed scheme" for collecting and redistributing) a deck of np cards into an n by p matrix and attempting to identify the row position of any elected card in the matrix after some fixed number of deals. Many special cases of this problem give rise to "magic tricks". In Section 1 of this paper, this problem will be precisely defined and the motivation for studying the "dynamical" properties of the function $x \mapsto \lfloor (x+r)/p \rfloor$ will be provided. In Section 2, a general formula for the tth iterate of the aforementioned function will be developed and used to determine the position of any card after the tth deal. Also, the cases in which the aforementioned function has a unique fixed point will be identified and a value (based on n and p) will be given for how many times the deck must be dealt to ensure that the selected card has reached the fixed position. Finally, in Section 3, the results of Section 2 will be extended to more "complicated" schemes in which the collecting and redistributing of the deck is based upon the definition of a periodic omega word in which all of the values are between 0 and $p-1$ inclusive.

1. Preliminaries and background on the problem

The Gergonne p-pile problem was first proposed by Gergonne in the early nineteenth century, and it is named in his honor. In general, one has a deck of np cards with one particular card in the deck selected. The problem involves distributing these cards, row by row, into an array of n rows and p columns and then re-collecting the cards column by column. This procedure of distribution and re-collection is repeated many times, and the problem is to keep track of the location of the one card that was selected in the beginning. Before addressing the details of the general problem, we will consider the specific example in which $p=3$ and $n=7$. This example is also of interest because it

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relates to a well-known magic trick with 21 cards, and so we will describe this magic trick as we carry out our illustration.

Assume we have a deck of 21 cards numbered 1 through 21. In order to select a particular card (the *target card*), a member of the audience is asked to choose a card at random from the deck and show it to the rest of the audience. Then, without the magician's having seen the target card, the audience member reinserts it in the deck and shuffles the deck well. It is now the magician's task to identify the target card. To this end, he deals the cards into an array of 3 columns and 7 rows, proceeding row by row from top to bottom and from left to right within each row. He then asks the audience which of the 3 columns contains the target card. Having been told which column contains the target card, the magician picks up the cards column by column and from top to bottom within each column, being careful that the column containing the target card is the column which he picks up *second*. Thus, the top card in the column containing the target card has become the eighth card from the top in his pile of cards, the second card in this column has become the ninth, etc. The magician now deals out the cards once again, starting at the top of his pile and working his way down, laying the cards out row by row and from left to right within each row into 3 columns and 7 rows as before. Again he asks which column contains the target card, and again he re-collects the cards column by column and from top to bottom within each column, picking up the column containing the target card second. For the third and final time, the magician deals out the cards as before and asks which column contains the target card. He then re-collects the cards column by column and from top to bottom within each column, picking up the column containing the target card second. The magician now deals the first ten cards, and, after making a suitable prediction, reveals that the eleventh card is the target card.

Let us actually track what happens to a specific card when we follow the scheme of the foregoing paragraph.

Example 1.1. Suppose that the initial configuration for a 21-card deck laid out in seven rows and three columns is

1	2	3
4	5	6
7	8	9
10	11	12
13	14	15
16	17	18
19	20	21

where the card labeled 1 is the card that has been selected by the member of the audience. When the magician asks which column contains the target card, the answer is column one. Assume that the magician picks up column three first, followed by column one (the target column), and then finally column two. Redistributing the cards, we obtain the following matrix:

3	6	9
12	15	18
21	**1**	4
7	10	13
16	19	2
5	8	11
14	17	20

Now, the audience member would tell the magician that column two contains the target card. Completing the final two re-collections and redistributions of the cards, we obtain the following configurations:

9	18	4		4	11	15
13	2	11		19	3	7
20	6	15		14	9	13
1	10	19	$\longrightarrow$	20	**1**	8
8	17	3		12	16	18
12	21	7		2	6	10
16	5	14		17	21	5

Note: For the second re-collection, the order in which the columns were picked up was 3,2,1. For the third, the order was 3,1,2.

If we re-collect the cards after the third redistribution (actually, the second will work as well), picking up the target column second, we see that card **1** will be the 11th card from the top of the deck when we distribute the cards for the final time. Thus, the *trick* (and hence, this example) is completed.

We will now talk about the general case of this problem. Assume we have a deck containing np cards dealt into an n-row by p-column matrix where n and p are both positive integers satisfying $n, p \geqslant 2$. Let x denote the current row position of the target card (hence, $1 \leqslant x \leqslant n$). We will let k represent the number of columns picked up before the target column when the cards are re-collected. In Section 3 of this paper, we will allow ourselves to pick up a *different number of columns* after each redistribution of the cards, but for now k will be fixed (i.e., we always pick up the same number of columns before the target column prior to each redistribution).

Since we always pick up k columns before the target column, it follows that there are $kn + x - 1$ cards before the target card whenever the cards are re-collected. Since we distribute the cards *row by row*, it follows that $(kn + x - 1)/p$ rows are filled before the target card is redealt. For instance, in Example 1.1 we have $n = 7$, $p = 3$, $k = 1$, and $x = 1$. Hence, there are $(7(1) + 1 - 1)/3 = 2\frac{1}{3}$ rows *filled* before we deal out the target card. Thus, in this example, the new row position of the target card after redistribution is row $\lfloor 2\frac{1}{3} \rfloor + 1 = 3$. It follows that, in general, given the current row position x of the target card, the new position of the target card after collection and redistribution is

$$\left\lfloor \frac{kn+x-1}{p} \right\rfloor + 1 = \left\lfloor \frac{x+(kn+p-1)}{p} \right\rfloor .$$

Letting $r = kn + p - 1$, we see that we want to study the dynamics of the function f which takes x to $\lfloor (x+r)/p \rfloor$.

As you can probably guess, we would like to determine the following information about f: (1) find a nice formula for the ℓth iterate of f (which represents the row position of the card after the ℓth redistribution); (2) determine all fixed points of f (there will be exactly one fixed point in many instances), and if there is only one fixed point y, determine whether it is a global attractor (i.e., whether for every $x \in \{1,\dots,n\}$ there exists an ℓ such that $f^{(\ell)}(x) = y$); and (3) determine how many iterations it takes an $x \in \{1,\dots,n\}$ to reach the fixed point.

You might believe at this point that f always has one unique fixed point no matter what k, n, and p are. However, this is not the case, as we shall see in the following example.

Example 1.2. Consider first the following 8-row by 3-column matrix of 24 cards in which **5** is the target card,

1	2	3
4	**5**	6
7	8	9
10	11	12
13	14	15
16	17	18
19	20	21
22	23	24

Using a distribution and re-collection pattern similar to the one used in Example 1.1, we obtain the following three matrices:

3	6	9		9	18	2		18	20	22
12	15	18		11	20	4		21	23	6
21	24	2		13	22	3		8	10	2
5	8	11	$\longrightarrow$	12	21	**5**	$\longrightarrow$	4	3	**5**
14	17	20		14	23	7		7	15	17
23	1	4		16	6	15		19	9	11
7	10	13		24	8	17		13	12	14
16	19	22		1	10	19		16	24	1

(For the first re-collection, the order in which the columns were picked up was 3,2,1. For the second, it was 3,1,2. For the third, it was 2,3,1.) Note that after the third (actually after the first as well) redistribution the card numbered **5** has reached the fourth row and will remain there. Thus, the card will be 12th from the top in the final deal of the cards.

In contrast, consider the initial layout

1	2	3
4	5	6
7	8	9
10	11	12
13	14	15
16	17	18
19	20	21
22	23	**24**

Redistributing and re-collecting three times, we obtain

2	5	8		8	17	3		3	4	2
11	14	17		12	21	4		6	7	11
20	23	3		13	22	2		18	19	17
6	9	12	⟶	11	20	6	⟶	21	22	20
15	18	21		15	**24**	7		**24**	5	9
24	1	4		16	5	14		10	8	12
7	10	13		23	9	18		13	11	15
16	19	22		1	10	19		16	23	1

(For the first re-collection, the order in which the columns were picked up was 2,3,1. For the second, it was 3,1,2. For the third, it was 3,2,1.) Note that card **24** remains fixed in row five after the third (actually after the second as well) redeal. Hence, it would be the 13th card from the top of the deck when the cards are finally redistributed. Thus, for the magician, an 8×3 configuration in which one column is always picked up before the target column would *not* be a good one to use since he would not be sure whether the target card was in the 12th or 13th position after the third re-collection. Note that the function f which takes x to $\lfloor (x + (1(8) + 3 - 1))/3 \rfloor = \lfloor (x + 10)/3 \rfloor$ has *two fixed points*, namely 4 and 5.

In Section 2 of this paper, we will find a *nice* formula for the lth iterate of f. This formula will help us determine when f does indeed have a unique fixed point. Also, we will find how many iterations of f it takes for a value of x in $\{1, \dots, n\}$ to converge to a particular fixed point of f (given fixed values of n and p).

2. Analysis of the fixed column re-collection case

In this section, we will assume that when we recollect the cards k columns are always picked up before the target column. Hence, as was discussed in the previous section, $kn + x - 1$ cards are always picked up before the selected target card, and so in this case $r = kn + p - 1$. In Theorem 2.1, we shall derive an expression for the lth iterate of the function f (defined below). This will help give us an idea where the *potential* fixed points of f lie.

Definition 1. Let r and p be positive integers with $p \geqslant 2$. We define f to be the function

$$f(x, r, p) = \left\lfloor \frac{x+r}{p} \right\rfloor,$$

where x is any integer. When no ambiguity can arise, we will abuse notation by writing $f(x)$ in place of $f(x, r, p)$.

Theorem 2.1. *Let r and p be fixed positive integers with $p \geqslant 2$, and write $f(x) = \lfloor (x+r)/p \rfloor$. For $\ell \geqslant 1$, we have*

$$f^{(\ell)}(x) = \left\lfloor \frac{x}{p^\ell} + \frac{r}{p-1}\left(1 - \frac{1}{p^\ell}\right) \right\rfloor,$$

where $f^{(\ell)}(x)$ denotes the ℓth iterate of the function $f(x)$.

Proof. When $\ell = 1$, we have

$$\frac{x}{p^\ell} + \frac{r}{p-1}\left(1 - \frac{1}{p^\ell}\right) = \frac{x+r}{p},$$

and so the theorem holds in this case. We now proceed by induction. Assuming that the theorem holds for some positive integer ℓ, we see that

$$\begin{aligned} f^{(\ell+1)}(x) = f\left(\left\lfloor \frac{x}{p^\ell} + \frac{r}{p-1}\left(1 - \frac{1}{p^\ell}\right) \right\rfloor\right) &= \left\lfloor \frac{\left\lfloor \frac{x}{p^\ell} + \frac{r}{p-1}\left(1 - \frac{1}{p^\ell}\right) \right\rfloor + r}{p} \right\rfloor \\ &= \left\lfloor \frac{\left\lfloor \frac{x}{p^\ell} + \frac{r}{p-1}\left(1 - \frac{1}{p^\ell}\right) + \frac{r}{p-1}(p-1) \right\rfloor}{p} \right\rfloor \\ &= \left\lfloor \frac{x}{p^{\ell+1}} + \frac{r}{p-1}\left(1 - \frac{1}{p^{\ell+1}}\right) \right\rfloor. \end{aligned}$$

So the theorem holds for $\ell + 1$, and the proof follows by induction. $\square$

Theorem 2.2. *Let r, p, and f be the same as in Theorem 2.1. The value $\lim_{\ell \to \infty} f^{(\ell)}(x)$ exists for every x, and we have*

$$\lim_{\ell \to \infty} f^{(\ell)}(x) = \begin{cases} \lfloor r/(p-1) \rfloor - 1 & \text{if } (p-1) \mid r \text{ and } x < r/(p-1), \\ \lfloor r/(p-1) \rfloor & \text{otherwise.} \end{cases}$$

In particular, we notice that $\lfloor r/(p-1) \rfloor$ is always a fixed point of f and that $\lfloor r/(p-1) \rfloor - 1$ is a fixed point of f exactly when $(p-1) \mid r$. Moreover, these are the only two possible fixed points of f.

Proof. Consider the sequence of real numbers defined for $\ell \geqslant 1$ by

$$a_\ell = \frac{r}{p-1} + \frac{1}{p^\ell}\left(x - \frac{r}{p-1}\right).$$

From Theorem 2.1 we see that $f^{(\ell)}(x) = \lfloor a_\ell \rfloor$, and so our task is to calculate the limit of the sequence $\{\lfloor a_\ell \rfloor\}$. It is obvious that the limit of the sequence $\{a_\ell\}$ is $r/(p-1)$, and so we see that the limit of the sequence $\{\lfloor a_\ell \rfloor\}$, if it exists, will be equal to either $\lfloor r/(p-1) \rfloor$ or $\lfloor r/(p-1) \rfloor - 1$. We must now show that the limit does always exist and determine when each of these two possibilities holds. Notice that the sequence $\{a_\ell\}$ is monotone increasing or monotone decreasing according to whether $x < r/(p-1)$ or $x > r/(p-1)$, respectively. Since $r/(p-1)$ is the limit of the sequence $\{a_\ell\}$, we observe that the sequence $\{\lfloor a_\ell \rfloor\}$ will have the limit $\lfloor r/(p-1) \rfloor$ whenever $\{a_\ell\}$ is a monotone decreasing sequence or whenever $\{a_\ell\}$ is a monotone increasing sequence but the limiting value $r/(p-1)$ is not an integer. In case the sequence $\{a_\ell\}$ is monotone increasing and the limiting value $r/(p-1)$ is an integer, we see that the limitting value of $\{\lfloor a_\ell \rfloor\}$ is $\lfloor r/(p-1) \rfloor - 1$. Thus, the assertion of the theorem follows in every case except $x = r/(p-1)$. But this final case is immediate, and so we are done. □

Hence, in reference to the original card trick, we are assured of a unique fixed position for the card if and only if $p-1$ does not divide $kn + p - 1$ or in the two cases when $k = 0$ or $k = p-1$. Note that whenever $k = 0$, Theorem 2.2 predicts that the two fixed points will be at row "positions" 0 and 1. Since the minimum row position is of course 1, the fixed point 0 is not between 1 and n, and so even though it is a fixed point of the integer valued function f, it is not a fixed point of f restricted to the set $\{1, \ldots, n\}$. Similarly, when $k = p-1$, n is the only fixed point. Now, let us examine the rate at which the values converge to the fixed points. You can probably guess that it is logarithmic in n and p. The following theorem will state this more precisely.

Theorem 2.3. *Let r, p and f be as in Theorem 2.1. We have $f^{(\ell)}(x) = f^{(\ell-1)}(x)$ whenever $\ell > \log_p |x - r/(p-1)| + 1$.*

Proof. As in the proof of Theorem 2.2, let

$$a_\ell = \frac{r}{p-1} + \frac{1}{p^\ell}\left(x - \frac{r}{p-1}\right).$$

and recall that $f^{(\ell)}(x) = \lfloor a_\ell \rfloor$. Consider first the case in which $x > r/(p-1)$, hence, the case in which $\{a_\ell\}$ is a monotone decreasing sequence. In this case we see that the values of $\{\lfloor a_\ell \rfloor\}$ become constant as soon as a_ℓ is below the value $\lceil r/(p-1) \rceil$. That is to say, the values will become constant as soon as $a_\ell - r/(p-1) < (p-1-\tilde{r})/(p-1)$, where $\tilde{r}$ denotes the least non-negative integer which is congruent to r modulo

$p-1$. Inserting the definition of the a_ℓ and noticing that $(p-1-\tilde{r})/(p-1) \neq 0$, we see that this happens exactly when

$$\left(x-\frac{r}{p-1}\right)\left(\frac{p-1}{p-1-\tilde{r}}\right) < p^\ell.$$

Since $(p-1)/(p-1-\tilde{r}) \leqslant p-1$, we see that the values in the sequence $\{\lfloor a_\ell \rfloor\}$ become constant whenever $\ell > \log_p |x - r/(p-1)| + 1$.

Now, consider the case in which $x < r/(p-1)$, hence, the case in which $\{a_\ell\}$ is a monotone increasing sequence. This case is handled in essentially the same way as the last one, but a little care has to be taken since it is possible that $(p-1)|r$, in which event the limit of $\{\lfloor a_\ell \rfloor\}$ is $\lfloor r/(p-1) \rfloor - 1$ and not $\lfloor r/(p-1) \rfloor$. Suppose first that $(p-1)|r$. We see then that the values of $\{\lfloor a_\ell \rfloor\}$ become constant as soon as a_ℓ is above the value $\lfloor r/(p-1) \rfloor - 1$. Thus, the values will be constant as soon as $r/(p-1) - a_\ell \leqslant 1$. Using the definition of a_ℓ, we see immediately that this happens as soon as $\ell \geqslant \log_p |x - r/(p-1)|$. Now, suppose that $p-1 \nmid r$. We see then that the values of $\{\lfloor a_\ell \rfloor\}$ become constant as soon as a_ℓ is above the value $\lfloor r/(p-1) \rfloor$. Thus, the values will be constant as soon as $r/(p-1) - a_\ell \leqslant \tilde{r}/(p-1)$, where $\tilde{r}$ is as defined above. Using the definition of a_ℓ and the fact that, since $p-1 \nmid r$, $0 < (p-1)/\tilde{r} < p-1$, we see that the values will be constant as soon as $\ell \geqslant \log_p |x - r/(p-1)| + 1$.

Combining the results of the last two paragraphs, we see that, if $x \neq r/(p-1)$, the values of the sequence $\{\lfloor a_\ell \rfloor\}$ become constant whenever $\ell > \log_p |x - r/(p-1)|$. In the case $x = r/(p-1)$, the values of the sequence are always constant, and so this completes the proof. □

Corollary 2.4. *Let r, p, and f be as in Theorem 2.1. For any $\ell > \log_p |x - r/(p-1)| + 1$ we have*

$$f^{(\ell)}(x) = \begin{cases} \lfloor r/(p-1) \rfloor - 1 & \text{if } (p-1)|r \text{ and } x < r/(p-1), \\ \lfloor r/(p-1) \rfloor & \text{otherwise.} \end{cases} \tag{2.1}$$

In particular, if $1 \leqslant x \leqslant n$ and $r = kn + p - 1$ where $0 \leqslant k \leqslant p-1$, then we see that Eq. 2.1 holds whenever $\ell > \log_p n + 1$.

We now give some specific examples.

Example 2.5. Assume we have a deck of 280 cards which we deal into a 40 row by 7 column matrix (i.e., $n = 40$ and $p = 7$). We assume that we always pick up 4 columns before the target column (thus $k = 4$). Hence it follows that $p - 1 = 6$ and $r = kn + p - 1 = 166$. Since $p-1$ does not divide r, the function f from Corollary 2.4 has a unique fixed point of 27. It will take less than or equal to 3 iterations of f for any value between 1 and 40 inclusive to reach the fixed point of 27 (since $3 > \log_7 40 + 1$).

Example 2.6. Assume that $n=40$, $p=7$ and $k=3$ (i.e., we now pick up 3 columns prior to the target column rather than 4). Then $r=kn+p-1=126$ and thus is divisible by 6. Hence, this function has two fixed points, $r/(p-1)-1=20$ and $r/(p-1)=21$. The upper bound for the convergence rate does not depend on k and thus remains fixed at 3. Note that if you were trying to perform a "magic trick", these would not be good values to select since there are two fixed positions rather than only one unique fixed position for the selected card after 3 iterations.

In Section 3 of this paper, we shall extend the results of Section 2 so that we need not pick up the same fixed number of columns every time we re-collect the deck of cards.

3. Analysis of the periodic column re-collection case

In this section we will generalize the results of Section 2 of this paper. The motivation for this section will be made clear by the following example. Assume we have a deck of 210 cards initially laid out in a matrix of 35 rows and 6 columns. Suppose the magician wants to pick up 2 columns before the target column on the first re-collection, 3 columns on the second re-collection, and 4 columns before the target column on the third re-collection. Further, suppose that he wants to repeat this pattern of how many columns to pick up indefinitely. This motivates the question of whether the results of Section 2 can be generalized in instances such as this one. We call this the *periodic column re-collection case* since we can think of the re-collection pattern as the periodic sequence or *omega word* $\Omega=(234)^\omega$ over the three digits or *alphabet* $\{2,3,4\}$.

More generally, assume we have a deck containing np cards dealt into a matrix with n rows and p columns. Assume the re-collection pattern is periodic of length $m\geqslant 1$ (i.e., $\Omega=(k_1\cdots k_m)^\omega$, where $k_1,\ldots,k_m\in\{0,\ldots,p-1\}$). From Sections 1 and 2 we know that if we are in row position x and pick up k_i columns before the column containing the target column, then the new row position is given by $\lfloor (x-r_i)/p\rfloor$, where $r_i=k_in+p-1$. As in Definition 1, we will denote by $f(\cdot,r_i,p)$ the function which takes x to $\lfloor (x+r_i)/p\rfloor$. The composition of the m functions for $i=1,\ldots,m$ will be written

$$f(x,(r_1,\ldots,r_m),p)=f(f(\cdots(f(x,r_1,p))\cdots),r_{m-1},p),r_m,p).$$

We now deduce a lemma which simplifies the form of $f(x,(r_1,\ldots,r_m),p)$.

Lemma 3.1. *Let m and p be positive integers with $p\geqslant 2$, and let $r_1,\ldots,r_m$ be positive integers. With $f(x,(r_1,\ldots,r_m),p)$ as above, we have*

$$f(x,(r_1,\ldots,r_m),p)=f\left(x,\sum_{i=1}^{m} r_i p^{m-i},p^m\right).$$

Proof. The proof is by induction on m. Notice that the case $m=1$ is immediate. Now, suppose that the lemma holds for some $m \geqslant 1$. Then we see that, for any positive integer r_{m+1}, we have

$$\begin{aligned}
f(x,(r_1,\dots,r_{m+1}),p) &= \left\lfloor \frac{f(x,(r_1,\dots,r_m),p)+r_{m+1}}{p} \right\rfloor \\
&= \left\lfloor \frac{f\left(x, \sum_{i=1}^{m} r_i p^{m-i}, p^m\right) + r_{m+1}}{p} \right\rfloor \\
&= \left\lfloor \frac{\left\lfloor \frac{x+\sum_{i=1}^{m} r_i p^{m-i} + r_{m+1} p^m}{p^m} \right\rfloor}{p} \right\rfloor \\
&= f\left(x, \sum_{i=1}^{m+1} r_i p^{m-i}, p^{m+1}\right),
\end{aligned}$$

and so the lemma follows. □

With this lemma in hand, we can easily generalize the results of Section 2 to the case of periodic column re-collection. Throughout the remainder of this section, we fix positive integers $r_1,\dots,r_m$ and $p \geqslant 2$ and write $F(x)$ for $f(x,(r_1,\dots,r_m),p)$.

Theorem 3.2. *For ease of notation, let us write* $L=\sum_{i=1}^{m} r_i p^{m-i}/(p^m-1)$. *Then for any* $\ell > 1/m \log_p |x-L|+1$ *we have*

$$F^{(\ell)}(x)=\begin{cases} \lfloor L \rfloor - 1 & \text{if } L \in \mathbb{Z} \text{ and } x<L, \\ \lfloor L \rfloor & \text{otherwise.} \end{cases} \tag{3.1}$$

In particular, if $1 \leqslant x \leqslant n$ *and, for every* i, $r_i = k_i n + p - 1$ *where* $1 \leqslant k_i \leqslant p-1$ *and* $n \geqslant 2$, *then we see that Eq.* 3.1 *holds whenever* $\ell > (1/m)\log_p n + 1$.

Proof. This follows immediately from Lemma 3.1, Theorem 2.4, and the fact that $\log_{p^m} y = (1/m)\log_p y$. □

Remark 1. We see that Theorem 3.2 tells us that, when $1 \leqslant x \leqslant n$ and r_i is of the form $k_i n + p - 1$ for some $0 \leqslant k_i \leqslant p-1$ and some $n \geqslant 2$, we have $F^{(\ell)}(x) = F^{(\ell+1)}(x)$ whenever $\ell > (1/m)\log_p n + 1$. Remember however that this denotes the number of times that we have to iterate the composition of the m functions $f_{r_1},\dots,f_{r_m}$ in order to arrive at a fixed point. Hence, we must actually deal our deck of cards at least $m\lceil (1/m)\log_p n + 1 \rceil$ times.

4. Final comments and acknowledgements

Please note that the analysis in Section 3 of this paper can be extended to the ultimately-periodic case (i.e., the case in which the control word for the recollection pattern is of the form $\Omega = uv^{\omega}$ where u and v are strings over $\{0, \ldots, p-1\}$). We would like to thank Leon Harkleroad for his helpful comments that aided in the completion of this paper. For more on the origins of the Gergonne p-pile problem, see chapter 11 of [1]. If you want to examine specific cases of this problem which yield good "magic tricks", see [2].

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