$U_q(gl(m|1))$ and canonical bases

Sean Clark
Northeastern University / Max Planck Institute for Mathematics

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QUANTUM ALGEBRAS AND CANONICAL BASES

$\mathbf{U}_q(\mathfrak{g})$: algebra over $\mathbb{Q}(q)$ coming from root data of simple Lie algebra $\mathfrak{g}$.

($\sim 1990$) Lusztig and Kashiwara: miraculous bases for $\mathbf{U}_q^-(\mathfrak{g}) = \mathbf{U}_q(\mathfrak{n}^-)$

(CB1) $B$ is a $\mathbb{Z}[q, q^{-1}]$-basis of the integral form $\mathbb{A}\mathbf{U}_q^-(\mathfrak{g})$;

(CB2) For any $b \in B$, $\bar{b} = b$ ($\bar{\cdot}$ is natural involution $q \mapsto q^{-1}$);

(CB3) $\mathcal{PBW} \rightarrow B$ is $q\mathbb{Z}[q]$-unitriangular for any PBW

(CB4) $B$ induces a basis on suitable modules

Connections to geometry, combinatorics, categorification, . . .
QUANTUM SUPERALEGEBRAS

Question: what if \( \mathfrak{g} \) is a Lie superalgebra?

E.g. \( \mathfrak{gl}(m|n) \): linear maps on vector superspace \( \mathbb{C}^{m|n} \).

Super complications:

- Different simple roots \( \Rightarrow \) Different \( \mathbf{U}^- \) (in general)

  **Workaround:** Work with a standard choice, e.g.
  \[ \alpha_1 = \epsilon_1 - \epsilon_2, \alpha_2 = \epsilon_2 - \epsilon_3, \ldots \]

- Finite-dimensional representations = not semisimple (in general)

  **Workaround:** Work with a nice subcategory, e.g. polynomial \( V(\lambda) \) \( \subset V_{\text{vec}} \)

- “Chirality” in parameter \( q \); e.g. \( \mathbf{U}_q(\mathfrak{gl}(m|n))_0 \cong \mathbf{U}_q(\mathfrak{gl}(m)) \otimes \mathbf{U}_{q^{-1}}(\mathfrak{gl}(n)) \)

  **Workaround:** Restrict rank; e.g. \( \mathfrak{gl}(m|1) \)
**Some Known Results**

Few examples of CBs known:

- easy small rank examples on $\mathfrak{u}^-$; e.g. $\mathfrak{gl}(1|1)$, standard $\mathfrak{gl}(2|1)$, $\mathfrak{osp}(1|2)$
- $\mathfrak{osp}(1|2n)$ and anisotropic Kac-Moody super [C-Hill-Wang, '13]
- Partial results for $\mathfrak{gl}(m|n)$, $\mathfrak{osp}(2|2n)$ [C-Hill-Wang, '13; Du-Gu '14]

Why partial?

**Problems for $\mathfrak{gl}(m|n)$:**

- Only considers standard simple roots case;
- In standard case, PBW basis $\rightarrow B = (\text{CB1}), (\text{CB2})$
- If $m > 1$ and $n > 1$, this basis is not canonical! (Depends on PBW)
- When $m = 1$ or $n = 1$, get pseudo-canonical basis (a signed basis)
**ROOT DATA FOR \( \mathfrak{g}l(m|1) \)**

\[
P = \left( \sum_{i=1}^{m} \mathbb{Z} \epsilon_i \right) \oplus \mathbb{Z} \epsilon_{m+1} \text{ with } (\epsilon_i, \epsilon_i) = (-1)^{p(\epsilon_i)}, \quad P^\vee, \langle \cdot, \cdot \rangle \text{ as usual.}
\]

\[
\Phi = \{ \epsilon_i - \epsilon_j \mid 1 \leq i \neq j \leq m+1 \} \text{ with simple roots } \Pi = \{ \alpha_i \mid i \in I \}
\]

Standard choice: \( \Pi_{\text{std}} = \{ \epsilon_1 - \epsilon_2, \ldots, \epsilon_m - \epsilon_{m+1} \} \)

Note:
- odd roots are isotropic; e.g. \((\epsilon_m - \epsilon_{m+1}, \epsilon_m - \epsilon_{m+1}) = 1 + -1 = 0; \)
- no odd simple reflection! (Still obvious \(S_{m+1}\) action: Weyl groupoid)
- different choices of simple roots may have different GCMs!

GCMs for \( m = 3 \):

\[
\begin{bmatrix}
2 & -1 & 0 \\
-1 & 2 & -1 \\
0 & -1 & 0
\end{bmatrix}, \quad \begin{bmatrix}
2 & -1 & 0 \\
-1 & 0 & 1 \\
0 & 1 & 0
\end{bmatrix}
\]

standard choice \( \Pi = \{ \epsilon_1 - \epsilon_2, \epsilon_2 - \epsilon_4, \epsilon_4 - \epsilon_3 \} \)
**U_q(gl(m|1))**

Fix a choice of $\Pi$. We define $U = U_q(\Pi) = \mathbb{Q}(q) \langle E_i, F_i, q^h \mid i \in I, h \in P^\vee \rangle$

subject to *usual* relations: e.g.

$$q^h E_i q^{-h} = q^{\langle h, \alpha_i \rangle} E_i, \quad [E_i, F_j] = E_i F_j - (-1)^{p(i)p(j)} F_j E_i = \delta_{ij} \frac{q^{h_i} - q^{-h_i}}{q - q^{-1}}$$

and both *usual* and *unusual* Serre relations: if $p(i) = 1$,

$$E_i^2 = F_i^2 = 0, \quad [E_{i-1}, [E_i, [E_{i+1}, E_i]_q]_q]_q = 0$$

This has standard structural features

(integral form, triangular decomposition, bar-involution, ...)

**NOTE:** Different $\Pi$ yield *different* $U^- = \mathbb{Q}(q) \langle F_i \mid i \in I \rangle$ in general!
**EXAMPLES FOR** \( m = 2 \)

Let \( A = \begin{bmatrix} 2 & -1 \\ -1 & 0 \end{bmatrix} \), \( B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \), both GCMs for \( \mathfrak{gl}(2|1) \).

\( \mathbb{U}^{-}(A) \) has generators \( F_1, F_2 \) subject to the relation

\[
F_1^2 F_2 + F_2 F_1^2 = (q + q^{-1}) F_1 F_2 F_1; \quad F_2^2 = 0
\]

So \( \mathbb{U}^{-}(A) \) has basis of form \( F_x^a F_1^x \) (where \( a \in \mathbb{Z}_{\geq 0} \) and \( x, y \leq 1 \))

\( \mathbb{U}^{-}(B) \) has generators \( F_1, F_2 \) subject to the relations

\[
F_1^2 = F_2^2 = 0.
\]

So \( \mathbb{U}^{-}(B) \) has basis e.g. \( F_1 F_2 F_1 F_2 F_1 F_2 \).

Different GCMs typically yield non-isomorphic half-quantum groups!
THE GOAL

Goal: Construct (non-signed!) canonical basis for $\mathcal{U}^- = U_q(\Pi_{\text{std}})$

Strategy:
1. Construct crystal on $\mathcal{U}^-$ using Benkart-Kang-Kashiwara;
2. Globalize using pseudo-canonical basis
   $\Rightarrow$ get canonical $B$ satisfying (CB1), (CB2), and (CB4) in many cases
3. Prove (CB3) using a braid group action.
**U-MODULES AND CRYSTALS**

As usual, can consider finite-dimensional weight representations.

\( \lambda \in P^+ \): weights in \( P \) which are \( \mathfrak{gl}(m) \)-dominant

\( K(\lambda) \): induced module from \( U_q(\mathfrak{gl}(m)) \)-rep \( V_{\mathfrak{gl}(m)}(\lambda) \)

\( V(\lambda) \): simple quotient

Crystal basis is a pair \((L, B)\) where

- \( L \) is a lattice over \( \mathcal{A} \subset \mathbb{Q}(q) \) (no poles at 0)
- \( B \) is a basis of \( L/qL \leftrightarrow \) nodes on a colored digraph
- \( E_i, F_i \leadsto \) operators \( \tilde{e}_i, \tilde{f}_i \) on \( L \) which, mod \( q \), move along the arrows

**Theorem (Benkart-Kang-Kashiwara, Kwon)**

Let \( \lambda \in P^+ \). Then \( K(\lambda) \) has a crystal basis \((L^K(\lambda), B^K(\lambda))\). If additionally \( \lambda \) is a polynomial, \( V(\lambda) \) admits a crystal basis \((L(\lambda), B(\lambda))\), combinatorially realized by super semistandard Young tableaux.
**Super Semistandard Tableaux for \( gl(2|1) \)**

Super alphabet: \( S = \{1, 2\} \cup \{\bar{3}\} \)

Semistandard tableaux in \( S \) means a Young diagram colored by \( S \) such that:
- even (odd) letters are (strictly) increasing along rows
- odd (even) letters are (strictly) increasing along columns

“Read” tableaux from right-to-left and top-to-bottom to get element of \( V_{\text{std}}^{\otimes t} \)

The \( \tilde{f}_i, \tilde{e}_i \) act via the “tensor product rule”

Vector representation:

\[
\begin{array}{c}
1 \xrightarrow{\alpha_1} 2 \xrightarrow{\alpha_2} \bar{3}
\end{array}
\]

\[
\begin{array}{ccc}
1 & 1 & 2 \\
\bar{3} & \bar{3} & \bar{3}
\end{array}
\]

\Rightarrow

\[
\begin{array}{ccccc}
2 & \otimes & 1 & \otimes & 1 & \otimes & \bar{3} & \otimes & \bar{3}
\end{array}
\]

Simple rule for odd root: Always apply \( \tilde{f}_2 \) to first 2 or \( \bar{3} \) encountered.
**Crystal for $\mathfrak{U}^-$**

**Theorem (C)**

$\mathfrak{U}^- = \mathfrak{U}^- (\Pi_{\text{std}})$ admits a crystal basis $\mathcal{B}$ which is compatible with those on modules. Moreover, these crystals “globalize” to a basis satisfying (CB1), (CB2), and (CB4) for

- the half-quantum enveloping algebra $\mathfrak{U}^-;$
- the Kac modules $K(\lambda)$ for any $\lambda \in P^+$;
- the simple modules $V(\lambda)$ for polynomial $\lambda \in P^+$.

Crystal for $\mathfrak{U}_q^-(\mathfrak{gl}(2|1))$

Crystal for $V(3\epsilon_1 + \epsilon_2)$
**Ideas in Proof**

- Simplify [BKK] results: for the $m \geq n = 1$ case,
  - no upper crystal part ($\leftrightarrow \mathfrak{gl}(n)$ part);
  - no fake highest weights: $\tilde{e}_i x = 0$ for all $i$ implies $x = v_\lambda$
  - not signed basis (odd operators never pass odd-colored boxes).

- Construct crystal inductively using (truncated) “grand loop” and [BKK].

- Characterize lattice and (signed) basis with bilinear form.

- Construct integral form of lattice in usual way.

- Existence of globalizations follows from pseudo-canonical basis
**Odd reflections**

To relate CB and PBW, want a braid group action (following Lusztig, Saito, Tingley).

Non-super: automorphism $T_i : U_q(\mathfrak{g}) \to U_q(\mathfrak{g})$ “lifting $s_i$ action on weights”

$$T_iU_{\nu} \subset U_{s_i(\nu)}$$

We want an analogue for super, but there is a

**Problem:** Odd isotropic simple root $\alpha_i \Rightarrow$ no odd reflection $s_i \in W$!
(There is a formal odd “reflection” in a Weyl groupoid.)

**Consequence:** No odd braid automorphism, but what about isomorphisms?
Perspective on Braids: Non-Super

Usual definition of braid action: automorphism $T_i : \mathbf{U}_q(\mathfrak{g}) \to \mathbf{U}_q(\mathfrak{g})$
e.g. for $\mathbf{U}_q(\mathfrak{sl}(3))$

$$T_2(F_1) = F_2F_1 - qF_1F_2 \in \mathbf{U}_q^-((\mathfrak{sl}(3)))_{s_2 \cdot \alpha_1}$$

Interpretation: $T_i$ is weight-preserving translation between choices of simples

Conjugacy of Borels $\leftrightarrow$ presentation is “unique”; e.g.

$$\left\{ F_{\epsilon_1 - \epsilon_2}, F_{\epsilon_2 - \epsilon_3}, \ldots \right\} \quad \Pi = \{ \epsilon_1 - \epsilon_2, \epsilon_2 - \epsilon_3 \}$$

$$\left\{ F_{\epsilon_1 - \epsilon_3}, F_{\epsilon_3 - \epsilon_2}, \ldots \right\} \quad \Pi = \{ \epsilon_1 - \epsilon_3, \epsilon_3 - \epsilon_2 \} \text{ all essentially the same}$$

$$\left\{ F_1, F_2, \ldots \right\} \quad \Pi = \{ \alpha_1, \alpha_2 \}$$

Braid operator is automorphism sending e.g.

$$T_2(F_1) = F_2F_1 - qF_1F_2 \quad \Rightarrow \quad T_2(F_{\epsilon_1 - \epsilon_2}) = F_{\epsilon_3 - \epsilon_2}F_{\epsilon_1 - \epsilon_3} - qF_{\epsilon_1 - \epsilon_3}F_{\epsilon_3 - \epsilon_2}$$
**Perspective on Braids: Super**

Borels are non-conjugate in general
→ can’t just identify different presentations!

**Example:** for standard $U_q(gl(2|1))$, $F_1^2 \neq 0$ yet

\[
\text{“} T_2(F_1)\text{”}^2 = (F_2F_1 - qF_1F_2)^2 = 0. 
\]

**Solution:** really, $T_2 : U(s_2 \cdot \Pi_{\text{std}}) \rightarrow U(\Pi_{\text{std}})$ with

\[
T_2(F_{\epsilon_1-\epsilon_3}) = F_{\epsilon_2-\epsilon_3} \underbrace{F_{\epsilon_1-\epsilon_2}}_{F_2} - qF_{\epsilon_1-\epsilon_2} F_{\epsilon_2-\epsilon_3} \underbrace{F_1}_{F_1}
\]

where $F_{\epsilon_1-\epsilon_3} = F_1 \in U(s_2 \cdot \Pi_{\text{std}})$; note $F_{\epsilon_1-\epsilon_3}^2 = 0$. 
Braid isomorphisms

Theorem (C)

If $\Pi$ and $\Pi'$ are related via the “reflection” $s_i$, then there is an isomorphism

$$T_i : U_q(\Pi) \rightarrow U_q(\Pi').$$

These maps satisfy braid relations:
Braid Definition

Identify generators from each copy by their GCMs $X$ and $Y$:

\[
T_i(E_{X,j}) = \begin{cases} 
-(-1)^{p_Y(i)} K_{Y,i}^{-1} & \text{if } j = i; \\
E_{Y,j} E_{Y,i} - (-1)^{p_Y(i)p_Y(j)} q^{e_{Yij}} E_{Y,i} E_{Y,j} & \text{if } j = i \pm 1; \\
E_{Y,j} & \text{otherwise;}
\end{cases}
\]

\[
T_i(F_{X,j}) = \begin{cases} 
-(-1)^{p_Y(i)} E_{Y,i} K_{Y,i}^e & \text{if } j = i; \\
F_{Y,i} F_{Y,j} - (-1)^{p_Y(i)p_Y(j)} q^{-e_{Yij}} F_{Y,j} F_{Y,i} & \text{if } j = i \pm 1; \\
F_{Y,j} & \text{otherwise;}
\end{cases}
\]

\[
T_i(K_{X,j}) = \begin{cases} 
(-1)^{p_Y(i)} K_{Y,i}^{-1} & \text{if } j = i; \\
(-1)^{p_Y(i)p_Y(j)} K_{Y,i} K_{Y,j} & \text{if } j = i \pm 1; \\
K_{Y,j} & \text{otherwise;}
\end{cases}
\]
PBW

\( w_0 = s_{i_1} \ldots s_{i_t} \): reduced expression for the longest element of \( S_{m+1} \)

→ root vectors and a PBW basis:

\[
F_\beta = T_{i_1} \ldots T_{i_{r-1}}(F_{i_r})
\]

Given \( w_0 = s_{j_1} \ldots s_{j_t} \), want same \( \mathbb{Z}[q] \) lattice of PBWs:

► Reduce to braid relations, hence rank 2 computations;
► New phenomena: three rank 2 cases:
  ▶ \[
  \begin{pmatrix}
  2 & -1 \\
  -1 & 2 \\
  \end{pmatrix}
  \]: \( sl(3) \)-case, which is known.
  ▶ \[
  \begin{pmatrix}
  2 & -1 \\
  -1 & 0 \\
  \end{pmatrix}
  \]: standard \( gl(2|1) \)-case, which is easy.
  ▶ \[
  \begin{pmatrix}
  0 & 1 \\
  1 & 0 \\
  \end{pmatrix}
  \]: non-standard \( gl(2|1) \)-case, which is false, but doesn’t occur.

Theorem (C)

\[ B \equiv_q \mathcal{PBW}_{(i_1,\ldots,i_t)}, \text{ so (CB3) holds.} \]
**Non-standard rank 2**

The non-standard GCM for $\mathfrak{gl}(2|1)$ is \[
\begin{bmatrix}
0 & 1 \\
1 & 0 
\end{bmatrix}.
\]
The PBW bases are given by

\[
F_1^x(F_2F_1 + q^{-1}F_1F_2)^{(y)}F_2^z, \quad F_2^x(F_1F_2 + q^{-1}F_2F_1)^{(y)}F_1^z.
\]

Clearly not the same $\mathbb{Z}[q]$-span; in fact, also not the same $\mathbb{Z}[q^{-1}]$-span!
(Nevertheless, there is a sort of CB: $F_1^x(F_2F_1)^yF_2^z$.)

However, this case doesn’t bother the theorem: e.g.

$$w_0 = s_3s_2s_1s_3s_2s_3 = s_3s_2s_1s_2s_3s_2$$

Way to see this in general: “weight-preserving” $\Rightarrow F_{\beta_1+\beta_2} = F_{\beta_2}F_{\beta_1} - qF_{\beta_1}F_{\beta_2}$
**Example: Canonical Basis of $\mathfrak{gl}(3|1)$**

Strategy for constructing $\mathcal{B}$ from canonical basis for $\mathfrak{gl}(m)$:

1. Inductively build $\mathcal{B}$ for weights with larger $\alpha_m$ multiplicity
2. Multiplicity $\leq 2^m$, so done in finite number of steps.

**Theorem (C)**

For $x, y, z \in \{0, 1\}$ and $a, b, c \in \mathbb{Z}_{\geq 0}$, let $u = u(x, y, z, a, b, c) \in \mathcal{B}$ be the unique element equal to the PBW vector $F_3^x F_2^{a+y} F_1^{b+c+z} F_3^y F_2^{b+z} F_3^z$ modulo $q$. Then

$$u = \begin{cases} 
F_3^x F_2^{(a+y)} F_1^{(b+c+z)} F_3^y F_2^{(b+z)} F_3^z & \text{if } c \geq a, \\
F_3^x F_2^y F_1^{(b+z)} F_3^y F_2^{(a+b+z)} F_3^z F_1^{(c)} & \text{if } a > c \text{ and } y \leq x, \\
\sum_{t=0}^{b} (-1)^t \binom{a-c-1+t}{t} F_2^{(a+1+t)} F_1^{(b+c+z)} F_3 F_2^{(b+z-t)} F_3^z & \text{otherwise.}
\end{cases}$$
Tantalizing hints of positivity

For standard $\mathfrak{gl}(2|1)$, easy CB: $F_2^x F_1^{(a)} F_2^y$ for $x, y \in \{0, 1\}, a \in \mathbb{Z}_{\geq 0}$.

Can directly check lots of positivity, e.g:

$$F_2^x F_1^{(a)} F_2^w F_2^y F_1^{(b)} F_2^z = \begin{cases} 0 & \text{if } y = w = 1; \\ \left[ \begin{array}{c} a + b \\ a \end{array} \right] F_2^x F_1^{(a+b)} F_2^z & \text{if } y = w = 0; \\ \delta_{z,0} \left[ \begin{array}{c} a + b - 1 \\ b \end{array} \right] F_2^x F_1^{(a+b)} F_2 + \delta_{x,0} \left[ \begin{array}{c} a + b - 1 \\ a \end{array} \right] F_2 F_1^{(a+b)} F_2 & \text{o.w.} \end{cases}$$

For $\mathfrak{gl}(3|1)$, harder to check but no counterexamples so far!
FURTHER QUESTIONS

- Categorification
  - CB interpretation in terms of Khovanov-Sussan?
  - Categorified modules?
  - What about non-standard half-quantum enveloping algebras?

- CB for other Lie superalgebras
  - For algebra, plausible for “simply laced with isolated isotropics”
  - Ex: can get signed CB for hqg associated to the Dynkin
    \[\begin{array}{ccc}
    \times & \circ & \cdots & \circ & \times \\
    \text{iso} & & & \text{iso}
    \end{array}\]
  - For modules, unclear: for example, crystals=disconnected in general!

- CB for standard \(\mathfrak{gl}(2|2)\) and beyond?
  - “chirality” is key difficulty; \(U_q(\mathfrak{gl}(m|n))_0 \cong U_q(\mathfrak{gl}(m)) \otimes U_{q^{-1}}(\mathfrak{gl}(n))\)
  - seems (CB3) is not correct condition!
THANKS FOR YOUR ATTENTION!

Some references:

- S. C., *Canonical bases for the quantum enveloping algebra of \( \mathfrak{gl}(m|1) \) and its modules*, arXiv:1605.04266
- J. Du and H. Gu, *Canonical bases for the quantum supergroups \( U(\mathfrak{gl}_{m|n}) \)*, Math. Zeit. 281, 631-660
- M. Khovanov, *How to categorify one-half of quantum \( \mathfrak{gl}(1|2) \)*, arXiv:1007.3517.
**Why not canonical?**

The standard GCM for $\mathfrak{gl}(2|2)$ is

\[
\begin{bmatrix}
2 & -1 & 0 \\
-1 & 0 & 1 \\
0 & 1 & -2
\end{bmatrix}.
\]

Let $U^-$ be the half-quantum group associated to this GCM. Depending on the order, the height 2 root vectors are either:

\[
F_{12} = F_2 F_1 - q F_1 F_2 \quad \text{(if $\alpha_1 < \alpha_2$)}; \quad F_{21} = F_1 F_2 - q F_2 F_1 \quad \text{(if $\alpha_2 < \alpha_1$)},
\]

\[
F_{23} = F_3 F_2 - q^{-1} F_2 F_3 \quad \text{(if $\alpha_2 < \alpha_3$)}; \quad F_{32} = F_2 F_3 - q^{-1} F_3 F_2 \quad \text{(if $\alpha_3 < \alpha_2$)}.
\]

With respect to $\mathbb{Z}[q]$, bar-invariant basis elements are

\[
\mathcal{B}_{\alpha_2 < \alpha_3} = \{F_2 F_3, F_3 F_2 - (q + q^{-1}) F_2 F_3 = F_{23} - q F_2 F_3\}
\]

\[
\mathcal{B}_{\alpha_3 < \alpha_2} = \{F_3 F_2, F_2 F_3 - (q + q^{-1}) F_3 F_2 = F_{32} - q F_3 F_2\}
\]

(If we take $\mathbb{Z}[q^{-1}]$ lattice, similar problem for $\alpha_1 + \alpha_2$)
**The curious case of "gl(1|1|1)"**

\[ \mathcal{U}_q^- (\{ \epsilon_1 - \epsilon_3, \epsilon_3 - \epsilon_2 \}) \cong \mathbb{Q}(q) \langle F_1, F_2 \rangle / (F_1^2, F_2^2) \]

\[ \mathbb{A} \mathcal{U}_q^- = \bigoplus \mathbb{Z}[q, q^{-1}] F_i^x (F_i F_j)^y F_j^z \text{ where } x, z \in \{0, 1\}, y \in \mathbb{Z}_{\geq 0} \]

Satisfies (CB1), (CB2), and (CB3) for unnormalized PBW \( F_k (F_\ell F_k + q F_k F_\ell)^y F_\ell^z \)

Moreover, has some compatibility with finite-dimensional simple modules: \( V(a \epsilon_1 + b \epsilon_2 + c \epsilon_3) \) is spanned by the non-zero \( F_i^x (F_i F_j)^y F_j^z \nu_\lambda \), with the linear dependence

\[ [a + c] (F_1 F_2)^{a-b} = [-b - c] (F_2 F_1)^{a-b} \]
THE CONJECTURAL $\mathfrak{gl}(1|2|1)$ CRYSTAL

$(\ast$ repeats$)$

$(\ast$ repeats$)$ — $H$ — $(\ast$ repeats$)$

$U\mathfrak{q}(\mathfrak{g}l(m|1))$ and canonical bases