Do Super Cats Make Odd Knots?

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What is a knot?

(The unknot)

(The Trefoil Knot)
WHAT IS A KNOT?

(Knots = \{ S^1 \hookrightarrow \mathbb{R}^3 \} / isotopy)

2D projection (avoiding triple intersections)

(The unknot)

(The Trefoil Knot)
**What is a knot?**

- **Knots** = \{ \mathbb{S}^1 \hookrightarrow \mathbb{R}^3 \} / isotopy
- 2D projection (avoiding triple intersections)
- Knots are isotopic iff projections equivalent under planar isotopy + Reidemeister moves
- Useful tool for distinguishing knots: invariants!
**Jones Polynomial and Khovanov Homology**

**Example (V. Jones, 1984)**

Given a knot (or link) diagram $D$, there is a Laurent polynomial $J_D = J_D(q)$ that is an invariant of knots. 

- $D = \bigcirc$ has $J_D = q + q^{-1}$.

- $D = \Bigcirc \bigcirc$ has 
  
  $J_D = -q^{-9} - q^{-7} + q^{-5} + 2q^{-3} + q^{-1}$.

Thus the trefoil is not the unknot!
JONES POLYNOMIAL AND KHovanov Homology

Example (V. Jones, 1984)
Given a knot (or link) diagram $D$, there is a Laurent polynomial $J_D = J_D(q)$ that is an invariant of knots. $D = \bigcirc$ has $J_D = q + q^{-1}$.

Example (Khovanov, 2000)
For a knot diagram $D$, construct complex $[D]$ of graded v.s.$/k$, subject to rules similar to Jones polynomial:

$$[\bigcirc] = 0 \to k[1] \oplus k[-1] \to 0 \quad "\Rightarrow" \quad q + q^{-1}$$

Khovanov Homology (KH) is the homology of this complex. The graded Euler characteristic of KH = Jones polynomial!
**Representation Theory**

Example (Reshetikhin-Turaev, late 1980’s)

Knots can be encoded in a category $\mathcal{TAN}$ of *tangles*. Given a “nice” Hopf algebra $H$ and module $V$, can find a functor from $\mathcal{TAN}$ to $H$-$\text{REP}$. This defines an operator invariant of the knot.

Special Case:
The quantum group $U_q(\mathfrak{sl}_2)$ is a “nice enough” Hopf algebra. This procedure with simple 2-dim module yields a map $\mathbb{Q}(q) \to \mathbb{Q}(q)$. Evaluation at 1 is the Jones polynomial!
CATEGORIFICATION

Both examples are categorifications:

(1-cat) $\text{KH} \xrightarrow{\chi} U\text{-mod} \xrightarrow{F} \text{Jones}$

(0-cat) $\text{Jones}$

Linked via categorified quantum groups (for all colored invariants)

Question: Can we find similar "explanation" for $\text{OKH}$?

Conjecture: Yes, with quantum osp$(1|2n)$ (Lie superalgebra)
Categorification

Both examples are categorifications:

(2-cat) \[ \hat{U} \text{-mod} \]

(1-cat) \[ KH \quad U \text{-mod} \]

(0-cat) \[ \text{Jones} \]

Linked via categorified quantum groups (for all colored invariants)
CATEGORIZATION

Both examples are categorifications:

(2-cat) \[ \mathcal{U} \text{-mod} \]
\[ W \quad K \]
\[ \mathcal{U} \text{-mod} \quad \text{OKH} \]
\[ \text{KH} \quad \text{Jones} \]

(1-cat) \[ \chi \quad F \]
\[ \text{KH} \quad \text{U-mod} \quad \text{Jones} \]

(0-cat) \[ \chi \quad \text{Jones} \]

- Linked via categorified quantum groups (for all colored invariants)
- Question: Can we find similar “explanation” for OKH?
**CATEGORIZATION**

Both examples are *categorifications*:

(2-cat) \[ \mathcal{U}\text{-mod} \]

\[ W \quad K \]

(1-cat) \[ \text{KH} \quad \mathcal{U}\text{-mod} \quad \text{OKH} \]

\[ \chi \quad F \]

(0-cat) \[ \text{Jones} \quad \mathcal{U}^s\text{-mod} \quad \text{Jones} \]

- Linked via *categorified quantum groups* (for all colored invariants)
- Question: Can we find similar “explanation” for OKH?
- Conjecture: Yes, with quantum $\mathfrak{osp}(1|2n)$ (Lie superalgebra)
WHAT IS $U^s$

Let $U^s = U_q(\mathfrak{osp}(1|2)) = \mathbb{Q}(q) \langle E, F, K, K^{-1}, J \rangle$ with rel’ns

\[
KK^{-1} = 1, \ KEK^{-1} = q^2 E, \ KFK^{-1} = q^{-2} F, \ EF + FE = \frac{JK - K^{-1}}{\pi - q - q^{-1}}
\]

\[J^2 = 1 \text{ and } J \text{ is central.}\]
WHAT IS $U^s$?

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**WHAT IS $U^s$**

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There are important module homomorphisms:

1. $R : X \otimes Y \cong Y \otimes X$ (R matrix) for any $X, Y$; satisfies braid rel’ns.

2. There is a simple 2-dim. module $V$.

$$\mathbb{Q}(q) \xrightarrow{\epsilon} V^* \otimes V \xrightarrow{\delta} \mathbb{Q}(q), \quad \mathbb{Q}(q) \xrightarrow{\epsilon'} V \otimes V^* \xrightarrow{\delta'} \mathbb{Q}(q)$$

$$\delta \circ \epsilon = q + \pi q^{-1} = \pi \delta' \circ \epsilon'$$
Knot Diagrams to Morphisms

Translate a knot diagram $D \xleftrightarrow{\sim} \text{a map } \mathbb{Q}(q) \to \mathbb{Q}(q) \text{ (constant)}$: 

\[ \begin{align*}
\alpha &= 1 \\
\beta &= 1 \\
\gamma &= 1 \\
\delta &= 1 \\
\epsilon &= 1
\end{align*} \]
Knot Diagrams to Morphisms
Translate a knot diagram $D \leftrightarrow$ a map $\mathbb{Q}(q) \rightarrow \mathbb{Q}(q)$ (constant):

- Cut diagram into simple pieces

Example:
$\sqrt{\pi} q - 1 = \sqrt{\pi} q + (\sqrt{\pi} q - 1)$

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Knot Diagrams to Morphisms

Translate a knot diagram $D \leftrightarrow$ a map $\mathbb{Q}(q) \rightarrow \mathbb{Q}(q)$ (constant):

- Cut diagram into simple pieces
- Translate each slice into a morphism

$\delta$

$1^* \otimes \delta \otimes 1$

$R \otimes R$

$1^* \otimes R \otimes 1$

$\epsilon \otimes \epsilon$

$1 = 1_V = \uparrow$

$1^* = 1_{V^*} = \downarrow$

$\sqrt{\pi^{\pm 1}} R = \times$

$\sqrt{\pi} \delta' = \bigcirc$

$\delta = \bigcirc$

$\sqrt{\pi^{-1}} \epsilon = \bigcirc$

$\epsilon' = \bigcirc$
Knot Diagrams to Morphisms

Translate a knot diagram $D \xleftrightarrow{}$ a map $\mathbb{Q}(q) \rightarrow \mathbb{Q}(q)$ (constant):

- Cut diagram into simple pieces
- Translate each slice into a morphism

$$\begin{align*}
1 &= 1_{\mathbb{V}} = \downarrow \\
1^* &= 1_{\mathbb{V}^*} = \downarrow \\
\sqrt{\pi \pm 1}R &= \times \\
\sqrt{\pi} \delta' &= \quad \delta = \\
\sqrt{\pi}^{-1} \epsilon &= \quad \epsilon' = \\
\end{align*}$$

- Compose and scale by $(\pi q)^\text{writhe}$

Then we get the Jones polynomial in the variable $\sqrt{\pi^{-1}}q$!

Example: $$\quad = \sqrt{\pi^{-1}}(q + \pi q^{-1}) = \sqrt{\pi^{-1}}q + (\sqrt{\pi^{-1}}q)^{-1} = \quad$$
Higher Rank and/or Colored Invariants

Theorem (C)

Let $K$ be a knot, $V(\lambda)$ a f.d. irrep. of $U^{\text{ns}} = U_q(\mathfrak{so}(1 + 2n))$ or $U^s = U_q(\mathfrak{osp}(1|2n))$, and $J^{s/\text{ns}}_K(q)$ the corresponding colored knot invariant. Then $J^s_K(q) = \sqrt{-1}^* J^{\text{ns}}_K(\sqrt{-1}q)$. 

Main idea in proof:

$\exists$ Complex isomorphism $\psi: \hat{U}^{\text{ns}} \sim \hat{U}^s$ with $\psi(q) = \sqrt{-1}q$.

$\psi$ induces a nice functor $\Psi$ on a rep category $\Psi X = \sqrt{-1}^* X \Psi$ where $X = \text{cup/cap/crossing}$

Conclusion:

$U^s$ does not give new invariants.

But it may lead to new odd knot homologies!
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- $\psi$ induces a nice functor $\Psi$ on a rep category
- $\Psi X = \sqrt{-1}^* X \Psi$ where $X =$ cup/cap/crossing

Conclusion: $U^{\text{s}}$ does not give new invariants. 😞
But it may lead to new odd knot homologies! 😊
CURRENT INTERESTS

- Construct an odd analogue of Webster’s construction. (An answer for the Jones polynomial would be nice!)

\[ \text{OKH} \quad \exists \quad \U^s\text{-mod} \quad \exists \quad \U\text{-mod} \quad \exists \quad \text{Jones} \]

- Studying these quantum groups at roots of unity.
- Further study of other types of quantum superalgebras.
- Categorification of quantum superalgebras and reps.
THANKS FOR YOUR ATTENTION!

Selected References:

S.C., *Quantum osp*(1|2n) knot invariants are the same as quantum so*(2n + 1)* invariants, arXiv:1509.03533


