Quantum supergroups and canonical bases

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What is a quantum group?

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Let $\mathfrak{g}$ be a semisimple Lie algebra (e.g. $\mathfrak{sl}(n), \mathfrak{so}(2n + 1)$).

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$U_q(\mathfrak{g})$ is the $\mathbb{Q}(q)$ algebra with generators $E_i, F_i, K^\pm_1$ for $i \in I$.

Various relations; for example,

- $K_i \approx q^{h_i}$, e.g. $K_i E_j K_i^{-1} = q^{\langle h_i, \alpha_j \rangle} E_j$
- quantum Serre, e.g. $F_i^2 F_j - [2]F_i F_j F_i + F_j F_i^2 = 0$
  (here $[2] = q + q^{-1}$ is a quantum integer)
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Some important features are:

- an involution $\bar{q} = q^{-1}$, $\bar{K}_i = K_i^{-1}$, $\bar{E}_i = E_i$, $\bar{F}_i = F_i$;
- a bar invariant integral $\mathbb{Z}[q, q^{-1}]$-form of $U_q(\mathfrak{g})$. 
Canonical basis and categorification

\[ U_q(n^-) \], the subalgebra generated by \( F_i \).
**Canonical basis and categorification**

$U_q(n^-)$, the subalgebra generated by $F_i$.

[Lusztig, Kashiwara]: $U_q(n^-)$ has a **canonical basis**, which

- is bar-invariant,
- descends to a basis for each h. wt. integrable module,
- has structure constants in $\mathbb{N}[q, q^{-1}]$ (symmetric type).

Relation to categorification:

$U_q(n^-)$ categorified by quiver Hecke algebras

[Khovanov-Lauda, Rouquier]: canonical basis $\leftrightarrow$ indecomp. projectives (symmetric type)

[Varagnolo-Vasserot].
CANONICAL BASIS AND CATEGORIZATION

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Relation to \textit{categorification}:

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        [Khovanov-Lauda, Rouquier]
  \item canonical basis \( \leftrightarrow \) indecomp. projectives (symmetric type)
        [Varagnolo-Vasserot].
\end{itemize}
Lie superalgebras

g: a Lie superalgebra (everything is $\mathbb{Z}/2\mathbb{Z}$-graded).
e.g. $\mathfrak{gl}(m|n)$, $\mathfrak{osp}(m|2n)$
LIE SUPERALGEBRAS

g: a Lie superalgebra (everything is \( \mathbb{Z}/2\mathbb{Z} \)-graded).
e.g. \( \mathfrak{gl}(m|n) \), \( \mathfrak{osp}(m|2n) \)

Example: \( \mathfrak{osp}(1|2) \) is the set of \( 3 \times 3 \) matrices of the form

\[
A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & c & d \\ 0 & e & -c \end{pmatrix} + \begin{pmatrix} 0 & a & b \\ -b & 0 & 0 \\ a & 0 & 0 \end{pmatrix}
\]

with the super bracket; i.e. the usual bracket, except
\[
[A_1, B_1] = A_1B_1 + B_1A_1.
\]

(Note: The subalgebra of the \( A_0 \) is \( \cong \) to \( \mathfrak{sl}(2) \).)
Quantized Lie superalgebras have been well studied (Benkart, Jeong, Kang, Kashiwara, Kwon, Melville, Yamane, ...)

Our question

Some potential obstructions are:

- Existence of isotropic simple roots: $(\alpha_i, \alpha_i) = 0$
- No integral form, bar involution (e.g. quantum $osp(1|2)$)
- Lack of positivity due to super signs

Experts did not expect canonical bases to exist!
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Influence of Categorification

- [KL,R] (’08): quiver Hecke categorify quantum groups
- [KKT11]: introduce quiver Hecke superalgebras (QHSA) (Generalizes a construction of Wang (’06))
- [KKO12]: QHSA’s categorify quantum supergroups (assuming no isotropic roots)
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- [HW12]: QHSA’s categorify quantum supergroups (assuming no isotropic roots)
Key Insight [HW]: use a parameter $\pi^2 = 1$ for super signs
e.g. a super commutator $AB + BA$ becomes $AB - \pi BA$
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Key Insight [HW]: use a parameter \( \pi^2 = 1 \) for super signs e.g. a super commutator \( AB + BA \) becomes \( AB - \pi BA \)

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There is a bar involution on \( \mathbb{Q}(q)[\pi] \) given by \( q \mapsto \pi q^{-1} \).

\[
[n] = \frac{(\pi q)^n - q^{-n}}{\pi q - q^{-1}}, \text{ e.g. } [2] = \pi q + q^{-1}.
\]

Note \( \pi q + q^{-1} \) has positive coefficients. (vs. \( -q + q^{-1} \))

(Important for categorification: e.g. \( F_i^2 = (\pi q + q^{-1})F_i^{(2)} \).)
\textbf{Anisotropic KM}

\[ I = I_0 \coprod I_1 \text{ (simple roots), parity } p(i) \text{ with } i \in I_p(i). \]

Symmetrizable generalized Cartan matrix \((a_{ij})_{i,j \in I}\):

- \(a_{ij} \in \mathbb{Z}, a_{ii} = 2, a_{ij} \leq 0;\)
- positive symmetrizing coefficients \(d_i (d_i a_{ij} = d_j a_{ji});\)
- (anisotropy) \(a_{ij} \in 2\mathbb{Z} \text{ for } i \in I_1;\)
- (bar-compatibility) \(d_i = p(i) \text{ mod } 2, \text{ where } i \in I_p(i)\)
EXAMPLES (FINITE AND AFFINE)

(●=odd root)

\[
\begin{align*}
\text{(osp}(1|2n)) & \quad \begin{array}{c}
\bullet \xrightarrow{} \circ \quad \cdots \quad \circ \xrightarrow{} \circ \\
\bullet \xrightarrow{} \circ \quad \cdots \quad \circ \xrightarrow{} \circ
\end{array} \\
\text{osp}(1|2n) & \quad \begin{array}{c}
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\end{align*}
\]
Finite type

The only finite type covering algebras have Dynkin diagrams

\[ \bullet \leftrightarrow \bigcirc \cdots \bigcirc \bigcirc \bigcirc \]
**FINITE TYPE**

The only finite type covering algebras have Dynkin diagrams

```
● ← ○ → ● · · · ○ → ○ ○ ○
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This diagram corresponds to

- the Lie superalgebra $\mathfrak{osp}(1|2n)$

[Zou98]
Finite type

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This diagram corresponds to
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- the Lie algebra $\mathfrak{so}(1 + 2n)$
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These algebras have similar representation theories.
- $\mathfrak{osp}(1|2n)$ irreps $\leftrightarrow$ half of $\mathfrak{so}(2n + 1)$ irreps.
FINITE TYPE

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These algebras have similar representation theories.

- $\mathfrak{osp}(1|2n)$ irreps $\leftrightarrow$ half of $\mathfrak{so}(2n + 1)$ irreps.
- $U_q(\mathfrak{osp}(1|2n))/\mathbb{C}(q) \leftrightarrow$ all of $U_q(\mathfrak{so}(2n + 1))$ irreps. [Zou98]
Rank 1

[CW]: $U_q(\mathfrak{osp}(1|2))/\mathbb{Q}(q)$ can be tweaked to get all reps.

$$EF - \pi FE = \frac{1K - K^{-1}}{\pi q - q^{-1}} \quad \text{or} \quad \frac{\pi K - K^{-1}}{\pi q - q^{-1}}$$

- even h.w.
- odd h.w.
**Rank 1**

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$$EF - \pi FE = \frac{\pi^h K - K^{-1}}{\pi q - q^{-1}} \quad (h \text{ the Cartan generator}) \quad (*)$$
[CW]: $\mathcal{U}_q(\mathfrak{osp}(1|2))/\mathbb{Q}(q)$ can be tweaked to get all reps.

$$EF - \pi FE = \frac{\pi^h K - K^{-1}}{\pi q - q^{-1}}$$ \hspace{1cm} (h the Cartan generator) \hspace{1cm} (*)

New definition: generators $E, F, K^\pm, J$, relations

$$J^2 = 1, \hspace{0.5cm} JK = KJ,$$

$$JEJ^{-1} = E, \hspace{0.5cm} KEK^{-1} = q^2 E, \hspace{0.5cm} JFJ^{-1} = F, \hspace{0.5cm} KFK^{-1} = q^{-2} F,$$

$$EF - \pi F_j E_i = \frac{JK - K^{-1}}{\pi q - q^{-1}};$$ \hspace{1cm} (*')

(If $h$ is the Cartan element, $K = q^h$ and $J = \pi^h$.)
**Definition of Quantum Covering Groups**

Let $A$ be a symmetrizable GCM. $U$ is the $\mathbb{Q}(q)[\pi]$-algebra with generators $E_i, F_i, K_i^{\pm 1}, J_i$ and relations

$$J_i^2 = 1, \quad J_iK_i = K_iJ_i, \quad J_iJ_j = J_jJ_i$$

$$J_iE_jJ_i^{-1} = \pi^{a_{ij}}E_j, J_iF_jJ_i^{-1} = \pi^{-a_{ij}}F_j.$$  

$$E_iF_j - \pi^{p(i)p(j)}F_jE_i = \delta_{ij}\frac{J_i^{d_i}K_i^{d_i} - K_i^{-d_i}}{(\pi q)^{d_i} - q^{-d_i}};$$

and others (super quantum Serre, usual $K$ relations).
DEFINITION OF QUANTUM COVERING GROUPS

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\[ J_i^2 = 1, \quad J_i K_i = K_i J_i, \quad J_i J_j = J_j J_i \]

\[ J_i E_j J_i^{-1} = \pi^{a_{ij}} E_j, J_i F_j J_i^{-1} = \pi^{-a_{ij}} F_j. \]

\[ E_i F_j - \pi^{p(i)p(j)} F_j E_i = \delta_{ij} \frac{J_i^{d_i} K_i^{d_i} - K_i^{-d_i}}{(\pi q)^{d_i} - q^{-d_i}}; \]

and others (super quantum Serre, usual $K$ relations).

Bar involution: $\bar{q} = \pi q^{-1}, \bar{K}_i = J_i K_i^{-1}, \bar{E}_i = E_i, \bar{F}_i = F_i$

Can also define a bar-invariant integral $\mathbb{Z}[q, q^{-1}, \pi]$-form!
RELATION TO QUANTUM (SUPER)GROUPS

By specifying a value of $\pi$, we have maps

$$U|_{\pi=-1} \quad \quad \quad U|_{\pi=1}$$

- $U|_{\pi=1}$ is a quantum group (forgets $\mathbb{Z}/2\mathbb{Z}$ grading).
- $U|_{\pi=-1}$ is a quantum supergroup.
REPRESENTATIONS

$X$: integral weights, $X^+$: dominant integral weights.

A weight module is a $U$-module $M = \bigoplus_{\lambda \in X} M_{\lambda}$, where

$$M_{\lambda} = \left\{ m \in M : K_i m = q^{\langle h_i, \lambda \rangle} m, \quad J_i m = \pi^{\langle h_i, \lambda \rangle} m \right\}.$$
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Example: $U_q(osp(1|2))$, $X = \mathbb{Z}$, $X^+ = \mathbb{N}$ and $M = \bigoplus_{n \in \mathbb{Z}} M_n$.

$$Jm = \pi^n m, \quad Km = q^n m \quad (m \in M_n)$$
Can define highest-weight (h.w.) and integrable (int.) modules.

**Theorem (C-Hill-Wang)**

For each $\lambda \in X^+$, there is a unique simple ("$\pi$-free") module $V(\lambda)$ of highest weight $\lambda$. Any ("$\pi$-free") h.wt. int. $M$ is a direct sum of these $V(\lambda)$.

($\pi$-free: $\pi$ acts freely)
Representations

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Example: $U_q(osp(1|2))$ has simple $\pi$-free modules $V(n)$, which are free $\mathbb{Q}(q)[\pi]$-modules of rank $n + 1$. (Like $\mathfrak{sl}(2)$!)

$$V(n) = \left[V(n)\big|_{\pi=1}\right] \oplus \left[V(n)\big|_{\pi=-1}\right]$$

$\dim_{\mathbb{Q}(q)} = n+1$ $\dim_{\mathbb{Q}(q)} = n+1$
Approaches to Canonical Bases

Two potential approaches to constructing a canonical basis:

- [Lusztig] using geometry
- [Kashiwara] algebraically using crystals ("$q = 0$")
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There are various crystal structures in modules:
- $\mathfrak{osp}(1|2n)$ [Musson-Zou] ('98)
- $\mathfrak{gl}(m|n)$ [Benkart-Kang-Kashiwara] ('00), [Kwon] ('12)
- for KM superalgebra with "even" weights [Jeong] ('01)

No examples of canonical bases.
Why believe?

No examples despite extensive study, experts don’t believe. Why *should* canonical bases exist?
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Because now we have

- a better definition of $U$ (all h. wt. modules $/\mathbb{Q}(q)$);
**Why believe?**

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Because now we have

- a better definition of $U$ (all h. wt. modules / $\mathbb{Q}(q)$);
- a good bar involution;
- a bar-invariant integral form;
Why believe?

No examples despite extensive study, experts don’t believe. Why should canonical bases exist?

Because now we have

- a better definition of $U$ (all h. wt. modules $/\mathbb{Q}(q)$);
- a good bar involution;
- a bar-invariant integral form;
- a categorical canonical basis.

This motivates us to try again generalizing Kashiwara.
**CRYSTALS**

We can define Kashiwara operators $\tilde{e}_i, \tilde{f}_i$.

Let $A \subset \mathbb{Q}(q)[\pi]$ be the ring of functions with no pole at $q = 0$.

$V(\lambda)$ is said to have a **crystal basis** $(\mathcal{L}, B)$ if

- $\mathcal{L}$ is a $A$-lattice of $V(\lambda)$ closed under $\tilde{e}_i, \tilde{f}_i$

and $B \subset \mathcal{L}/q\mathcal{L}$ satisfies

- $B$ is a $\pi$-basis of $\mathcal{L}/q\mathcal{L}$; (i.e. signed at $\pi = -1$: $B = B \cup \pi B$)
- $\tilde{e}_i B \subseteq B \cup \{0\}$ and $\tilde{f}_i B \subseteq B \cup \{0\}$;
- For $b \in B$, if $\tilde{e}_i b \neq 0$ then $b = \tilde{f}_i \tilde{e}_i b$.

As in the $\pi = 1$ case, the crystal lattice/basis is unique.
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As in the $\pi = 1$ case, the crystal lattice/basis is unique.
Theorem (C-Hill-Wang)

The pairs \((L(\lambda), B(\lambda))\) for \(\lambda \in X^+ \cup \{\infty\}\) are crystal bases. Moreover, there exist maps \(G: L(\lambda)/qL(\lambda) \to L(\lambda)\) such that \(G(B(\lambda))\) is a bar-invariant \(\pi\)-basis of \(V(\lambda)\).

We call \(G(B(\lambda))\) the canonical basis of \(V(\lambda)\).

\[ V(\lambda) \supseteq L(\lambda) = \sum A\tilde{f}_{i_1} \ldots \tilde{f}_{i_n} v_\lambda, \quad B(\lambda) = \left\{ \pi^\epsilon \tilde{f}_{i_1} \ldots \tilde{f}_{i_n} v_\lambda + qL(\lambda) \right\} \]

\((\lambda \in X^+ \cup \{\infty\}, V(\infty) = U^-)\)
**Canonical Basis**

We set

\[ V(\lambda) \supset L(\lambda) = \sum A \tilde{f}_{i_1} \ldots \tilde{f}_{i_n} v_{\lambda}, \quad B(\lambda) = \left\{ \pi^{\epsilon} \tilde{f}_{i_1} \ldots \tilde{f}_{i_n} v_{\lambda} + q L(\lambda) \right\} \]

\[(\lambda \in X^+ \cup \{\infty\}, V(\infty) = U^-)\]

**Theorem (C-Hill-Wang)**

*The pairs \((L(\lambda), B(\lambda))\) for \(\lambda \in X^+ \cup \{\infty\}\) are crystal bases.*

*Moreover, there exist maps \(G : L(\lambda)/qL(\lambda) \rightarrow L(\lambda)\) such that \(G(B(\lambda))\) is a bar-invariant \(\pi\)-basis of \(V(\lambda)\).*

We call \(G(B(\lambda))\) the canonical basis of \(V(\lambda)\).

\((\pi = -1: \text{first canonical bases for quantum supergroups!})\)
Main obstacle in proof

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Kashiwara’s construction of $G$ requires $\rho(\mathcal{L}(\infty)) \subset \mathcal{L}(\infty)$ where $\rho$ is an anti-automorphism of $U^-$. Super signs cause non-positivity problems $\Rightarrow$ usual proof fails.
MAIN OBSTACLE IN PROOF

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Kashiwara’s construction of $G$ requires $\rho(\mathcal{L}(\infty)) \subset \mathcal{L}(\infty)$ where $\rho$ is an anti-automorphism of $U^-$.

Super signs cause non-positivity problems $\Rightarrow$ usual proof fails.

New idea: a twistor (from work with Fan, Li, Wang [CFLW]).

\[
U^-|_{\pi=1} \otimes \mathbb{C} \xrightarrow{\cong} U^-|_{\pi=-1} \otimes \mathbb{C}
\]

which is almost an algebra isomorphism.

Good enough: the $\rho$-invariance at $\pi = 1$ transports to $\pi = -1$. 
**Why must the basis be signed?**

**Example:** $I = I_1 = \{i, j\}$ such that $a_{ij} = a_{ji} = 0$.

$$F_i F_j = \pi F_j F_i$$

Should $F_i F_j$ or $F_j F_i$ be in $B(\infty)$? No preferred canonical choice.
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Should $F_i F_j$ or $F_j F_i$ be in $B(\infty)$? No preferred canonical choice.

This is not a bad thing!

- A $\pi$-basis is an honest $\mathbb{Q}(q)$-basis (for $\pi$-free modules)!
- Categorically: represents “spin states” of QHSA modules.
CANONICAL BASES AND THE WHOLE QUANTUM GROUP

Can the canonical basis on $U^-$ be extended to $U$?
Not directly: $U^0$ makes such a construction difficult.
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The ‘right’ construction is to explode $U^0$ into idempotents.
(Beilinson-Lusztig-McPherson (type A), Lusztig)

$$1 \mapsto \sum_{\lambda \in X} 1_\lambda \text{ with } 1_\lambda 1_\eta = \delta_{\lambda, \eta} 1_\lambda, \quad K_i \mapsto \sum_{\lambda \in X} q^{\langle h_i, \lambda \rangle} 1_\lambda$$
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Not directly: $U^0$ makes such a construction difficult.

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$$1 \leadsto \sum_{\lambda \in X} 1_\lambda \text{ with } 1_\lambda 1_\eta = \delta_{\lambda,\eta} 1_\lambda, \quad K_i \leadsto \sum_{\lambda \in X} q^{\langle h_i, \lambda \rangle} 1_\lambda$$

$\dot{U}$ is the algebra on symbols $x 1_\lambda = 1_{\lambda + |x|} x$ for $x \in U, \lambda \in X$.

$x 1_\lambda = \text{projection to } \lambda\text{-wt. space followed by the action of } x$. 
**Rank 1**

\[ \mathcal{U}_q(\mathfrak{osp}(1|2)) \] is the algebra given by

**Generators:** \( E_1^n = 1_{n+2} E, \quad F_1^n = 1_{n-2} F, \quad 1_n \)

**Relations:** \( 1_n 1_m = \delta_{nm} 1_n, \quad (E_1^{n-2})(F_1^n) - (F_1^{n+2})(E_1^n) = [n]1_n \)
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\textbf{Theorem (C-Wang)}

$\mathcal{U}_q(\mathfrak{osp}(1|2))$ admits a canonical basis

\[ \hat{B} = \left\{ E^{(a)}1_nF^{(b)}, \pi^{ab}F^{(b)}1_nE^{(a)} \mid a + b \geq n \right\}. \]
**Rank 1**

$\mathcal{U}_q(osp(1|2))$ is the algebra given by

Generators: $E_1 = 1_{n+2}, F_1 = 1_{n-2}, 1_n$

Relations: $1_n 1_m = \delta_{nm} 1_n$, $(E_1)(F_1) - (F_1)(E_1) = [n] 1_n$

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We conjectured $\mathcal{U}_q(osp(1|2))$ admits a categorification, and Ellis and Lauda (’13) recently verified our conjecture.
Theorem (C)
\( \dot{U} \) admits a \( \pi \)-signed canonical basis generalizing the basis for \( U^- \).
For \( \pi = 1 \), this specializes to Lusztig’s canonical basis for \( U|_{\pi=1} \).
**Canonical Basis**

Theorem (C)

$\hat{U}$ admits a $\pi$-signed canonical basis generalizing the basis for $U^-$. For $\pi = 1$, this specializes to Lusztig’s canonical basis for $\hat{U}|_{\pi=1}$.

**Idea of proof (generalizing Lusztig):**
Consider modules $N(\lambda, \lambda') \to \hat{U}1_{\lambda-\lambda'}$ as $\lambda, \lambda' \to \infty$. 
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Construct suitable bar involution, canonical basis on \( N(\lambda, \lambda') \).
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The canonical basis is stable under the projective limit \( \Rightarrow \) induces a bar-invariant canonical basis on \( \hat{U} \).
FURTHER DIRECTIONS

- Construction of braid group action à la Lusztig
  - Forthcoming work with D. Hill
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- Canonical bases for other Lie superalgebras
  - $\mathfrak{gl}(m|1), \mathfrak{osp}(2|2n)$ using quantum shuffles [CHW3]
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- Categorification for covering quantum groups
  - Connection to odd link homologies (Khovanov)
  - Tensor modules?
  - Higher rank?
Some Related Papers

Lusztig, Canonical bases arising from quantized enveloping algebras, J. Amer. Math. Soc. 3 (1990), pp. 447–498


Ellis, Khovanov, Lauda, The odd nilHecke algebra and its diagrammatics, IMRN 2014 pp. 991–1062


Fan and Li, Two-parameter quantum algebras, canonical bases and categorifications, arXiv:1303.2429


Slides available at
http://people.virginia.edu/~sic5ag/

Thank you for your attention!