A canonical basis for covering quantum groups

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Quantum Groups

$q$: a generic parameter;
$\mathfrak{g}$: a Kac-Moody algebra with simple roots $\Pi = \{\alpha_i : i \in I\}$.
$U_q(\mathfrak{g})$ is the $\mathbb{Q}(q)$ algebra with generators $E_i, F_i, K_i^{\pm 1}$ for $i \in I$.
$U_q(n^-)$, the subalgebra generated by $F_i$.

$U_q(n^-)$ has many interesting properties, e.g.
- Lusztig-Kashiwara canonical basis;
- categorifications of Khovanov-Lauda and Rouquier;

$U_q(\mathfrak{g})$ admits a categorification for its modified form [L, KL, R].
**HALF QUANTUM SUPERGROUPS**

\( \mathfrak{g} \): an anisotropic Kac-Moody superalgebra with \( \mathbb{Z}/2\mathbb{Z} \)-graded simple roots \( \Pi = \Pi_0 \sqcup \Pi_1 = \{ \alpha_i : i \in I \} \)

\( \mathcal{U}_q(n^-) \): algebra generated by \( F_i \) satisfying *super* Serre relations. Was not expected to admit a canonical basis.

Super KLR= quiver Hecke superalgebras
(Ellis-Khovanov-Lauda in rank 1, Kang-Kashiwara-Tsuchioka independently defined the general construction)

[Hill-Wang] \( \mathcal{U}_q(n^-) \) is categorified by QHSA’s.
\( \Rightarrow \) It has a categorical canonical basis.

Is there an intrinsic canonical basis à la Lusztig, Kashiwara?
INSIGHT FROM [HW]

Anisotropic super and non-super are formally similar

Key Insight [HW]: use a parameter $\pi^2 = 1$ for super signs

▶ $\pi = 1 \leadsto$ non-super case.
▶ $\pi = -1 \leadsto$ super case.

There is a bar involution on $\mathbb{Q}(q)^\pi = \mathbb{Q}(q, \pi)/(\pi^2 - 1)$ given by

$$q \mapsto \pi q^{-1} \quad (\pi^2 = 1)$$

and quantum integers

$$[n] = \frac{(\pi q)^n - q^{-n}}{\pi q - q^{-1}}, \quad [n]!, \quad \begin{bmatrix} n \\ a \end{bmatrix} \in \mathbb{Z}[q, q^{-1}].$$

giving $U_q(n^-)$ a suitable bar-invariant integral form.
**Anisotropic KM**

We consider a KM superalgebra with GCM $A$ indexed by $I = I_0 \sqcup I_1$ (simple roots) and satisfying:

- $a_{ij} \in \mathbb{Z}$, $a_{ii} = 2$, $a_{ij} \leq 0$
- there exist positive symmetrizing coefficients $d_i$ ($d_i a_{ij} = d_j a_{ji}$)
- (anisotropy) $a_{ij} \in 2\mathbb{Z}$ for $i \in I_1$

We call these “of anisotropic type”. We will also impose:

- (bar-compatibility) $d_i \equiv_2 p(i)$
EXAMPLES

(●=odd root)

\[ \text{osp}(1|2n) \]
**Known Facts about KM Super**

Quantized Lie superalgebras have been well studied (Benkart, Jeong, Kang, Kashiwara, Kwon, Musson, Zou, ...)

Some key coincidences exist for anisotropic KM:

- $\mathfrak{osp}(1|2n)$ reps “=” half of $\mathfrak{so}(2n+1)$ reps (R.B. Zhang, Lanzmann)

- Over $\mathbb{C}(q)$, $U_q(\mathfrak{osp}(1|2n))$ miraculously has the missing reps. (Musson-Zou)

[CW]: $U_q(\mathfrak{osp}(1|2)) / \mathbb{Q}(q)$ can be tweaked to get all reps.

$$EF - \pi FE = \frac{K - K^{-1}}{\pi q - q^{-1}}$$  or  $$\frac{\pi K - K^{-1}}{\pi q - q^{-1}}$$

even h.w.  odd h.w.
DEFINITION [CHW1]

Let \( g \) be a KM superalgebra of anisotropic type, \( A \) its symmetrizable GCM.
Let \( U = U_q(g) \) be the \( \mathbb{Q}(q) \)-algebra with generators \( E_i, F_i, K_i^{\pm 1}, J_i \) such that

\[
J_i^2 = 1, \quad J_i K_i = K_i J_i, \quad J_i J_j = J_j J_i, \quad K_i K_j = K_j K_i,
\]

\[
J_i E_j J_i^{-1} = \pi a_{ij} E_j, \quad K_i E_j K_i^{-1} = q a_{ij} E_j,
\]

\[
J_i F_j J_i^{-1} = \pi^{-a_{ij}} F_j, \quad K_i F_j K_i^{-1} = q^{-a_{ij}} F_j,
\]

\[
E_i F_j - \pi^{p(i)p(j)} F_j E_i = \delta_{ij} \frac{J_i^{d_i} K_i^{d_i} - K_i^{-d_i}}{(\pi q)^{d_i} - q^{-d_i}},
\]

\[
\sum_{k=0}^{1-a_{ij}} (-1)^k \pi^{p(k;i,j)} E_i^{(1-a_{ij}-k)} E_j E_i^{(k)} = \sum_{k=0}^{1-a_{ij}} (-1)^k \pi^{p(k;i,j)} F_i^{(1-a_{ij}-k)} F_j F_i^{(k)} = 0,
\]

where \( p(k; i, j) = kp(i)p(j) + \frac{1}{2} k(k - 1)p(i) \).
**RANK 1**

For $U_q(\mathfrak{osp}(1|2))$

Generators: $E, F, K^{\pm 1}, J$

Relations:

$$J^2 = 1, \quad JK = KJ,$$

$$JEJ^{-1} = E, \quad KEK^{-1} = q^2 E,$$

$$JFJ^{-1} = F, \quad KFK^{-1} = q^{-2} F,$$

$$EF - \pi F_j E_i = \frac{JK - K^{-1}}{\pi q - q^{-1}};$$

(If $h$ is the Cartan element, $K = q^h$ and $J = \pi^h$.)

We call this covering quantum $\mathfrak{osp}(1|2)$ or $\mathfrak{sl}(2)$ ($\pi = 1$ case)
**FINITE TYPE**

The only finite type covering algebras have Dynkin diagrams

![Dynkin diagram](attachment:image.png)

This diagram corresponds to
- the Lie superalgebra $\mathfrak{osp}(1|2n)$
- the Lie algebra $\mathfrak{so}(1 + 2n)$

(NB. There is no "covering $\mathfrak{sl}(n)$" in this construction)
Structures in a Covering Quantum Group

$U$ has all the nice features you could hope for:

- $U = U^- \otimes U^0 \otimes U^+$;
- $U^-$ admits a nondegenerate bilinear form;
- there is a Hopf superalgebra structure (super sign $\mapsto \pi$);
- there is a bar involution ($K \mapsto JK^{-1}$);
- there is a quasi-$R$-matrix and Casimir-type operator;
Representations

Let $P (P^+)$ be the set of (dominant) weights of $\mathfrak{g}$.

A weight module is a $\mathcal{U}_q(\mathfrak{g})$-module $M = \bigoplus_{\lambda \in P} M^\lambda$, where

$$M^\lambda = \left\{ m \in M : K_i m = q^{\langle h_i, \lambda \rangle} m, \quad J_i m = \pi^{\langle h_i, \lambda \rangle} m \right\}.$$

We can define highest-weight and integrable modules as usual to obtain a semi-simple category $\mathcal{O}_{\text{int}}$.

Simple modules: $V(\lambda)$ for all $\lambda \in P^+$
(Same character as in classical case)
CRYSTALS

To construct a CB, we use the algebraic approach with crystals. Specifically, we construct a covering analogue for

- Kashiwara operators $\tilde{e}_i, \tilde{f}_i$;
- the crystal lattice;
- the action of the $q$-Boson algebra;
- the polarizations (= deformed Shapovalov forms);
- the tensor product rule;

Kashiwara’s grand loop argument can be extended to the covering case.

Moreover, this crystal basis admits globalization to a canonical basis.
**CANONICAL BASIS**

**Theorem (C-Hill-Wang)**

$U^- \text{ and the integrable modules admit compatible canonical bases.}$

Let $B$ be the canonical basis of $U^-$. 

- If $v_\lambda$ is the highest weight vector of $V(\lambda)$,

  $$bv_\lambda = 0 \text{ or is a CB element.}$$

- $B|_{\pi=1} =$ the Lusztig-Kashiwara CB

- $B$ is typically $\pi$-signed: $b \in B$ implies $\pi b \in B$.

**Example:** $a_{ij} = 0, p(i) = p(j) = 1$

$$F_iF_j = \pi F_j F_i$$

(Categorically: $M$ is not isomorphic to its parity shift $\Pi M$.)
**Modified Form**

Basic idea: \( 1 \sim \sum_{\lambda \in P} 1_{\lambda} \) with \( 1_{\lambda} 1_{\eta} = \delta_{\lambda, \eta} 1_{\lambda} \)

For \( x \in U \), let \(|x|\) be the weight.

\( \hat{U} \) is the algebra on symbols \( x1_{\lambda} = 1_{\lambda + |x|}x \) for \( x \in U, \lambda \in P \) satisfying

\[
(xy)1_{\lambda} = x1_{\lambda + |y|}y1_{\lambda}, \quad J_{\mu}K_{\nu}1_{\lambda} = \pi^{\langle \mu, \lambda \rangle}q^{\langle \nu, \lambda \rangle}1_{\lambda}
\]

Any weight \( U \)-module \( M \) is a \( \hat{U} \) module:
\( x1_{\lambda} \) acts as projection to \( M^{\lambda} \) followed by the \( U \)-action of \( x \).
SOME PROPERTIES

$\mathcal{U}$ has some additional useful properties:

- Automorphisms of $\mathcal{U}$ extend to $\mathcal{U}$;
- $\mathcal{U}_1^\lambda \overset{v.s.}{=} \mathcal{U}^- \otimes \mathcal{U}^+$;

Theorem (C.)

There is a non-degenerate symmetric bilinear form on $\mathcal{U}$ which:

- extends the form on $\mathcal{U}^-$;
- is invariant under our favorite maps;
- is a limit of polarizations;

For $\pi = 1$, this is Lusztig’s form on $\mathcal{U}$. 
**RANK 1**

$\hat{U}_q(\mathfrak{osp}(1|2))$ is the algebra given by

Generators: $E1_n = 1_{n+2}E$, $F1_n = 1_{n-2}F$, $1_n$

Relations: $1_n 1_m = \delta_{nm} 1_n$ and $EF1_n - FE1_n = [n]1_n$

Theorem (C-Wang)

$\hat{U}_q(\mathfrak{osp}(1|2))$ admits a canonical basis

$$\hat{B} = \left\{ E^{(a)}1_nF^{(b)}, \pi^{ab}F^{(b)}1_nE^{(a)} \mid a + b \geq n \right\}.$$ 

(In rank 1, the basis need not be $\pi$-signed)

Ellis and Lauda have categorified $\hat{U}_q(\mathfrak{osp}(1|2))$. 
CONSTRUCTING THE CB

- $\hat{U}1_{\lambda - \lambda'}$ projects "nicely" onto $N(\lambda, \lambda')$ (highest weight $\otimes$ lowest weight);
- $N(\lambda, \lambda')$ has a bar involution (Lusztig quasi-$\mathcal{R}$-matrix);
- $N(\lambda, \lambda')$ admits a CB (bar involution + CB on simples);
- The CB of $N(\lambda, \lambda')$ is compatible with $N(\lambda + \lambda'', \lambda'' + \lambda')$;

These facts allow us to build a basis for $\hat{U}$. 
Theorem (C)
\( \hat{U} \) admits a \( \pi \)-signed canonical basis generalizing the basis for \( U^- \). This basis is \( \pi \)-almost orthonormal under the bilinear form. For \( \pi = 1 \), this specializes to Lusztig’s canonical basis for \( \hat{U}|_{\pi=1} \).
SOME RELATED PAPERS

[C] Quantum supergroups IV. Modified form, forthcoming.

Slides available at
http://people.virginia.edu/~sic5ag/
Thank you for your attention!