My primary research interests lie in representation theory, and its connections with algebraic combinatorics, topology, geometry, and categorification. Specifically, I am most interested in the representation theory of quantum algebras, such as the quantized enveloping Lie (super)algebras, (q-)Schur (super)algebras, and (quiver) Hecke algebras. These algebras are deformations of classical algebras, which are related to a variety of topics, including knot invariants and topological quantum field theories, the combinatorics of Young tableaux and crystals, and higher representation theory.

My current research program involves the construction of *canonical bases* for quantum Lie superalgebras, and connecting these bases to categorifications appearing in higher representation theory. The canonical basis of a quantum algebra is a basis with a number of remarkable qualities. Recently, such bases have been realized to make deep connections with other areas of mathematics, often through higher representation theory (HRT) as pioneered by Chuang and Rouquier. In this setting, canonical bases typically encode an analogue of Kazhdan-Lusztig theory for the category.

On the other hand, Lie superalgebras are a generalization of Lie algebras motivated, in part, by supersymmetry in physics. These algebras are very similar to Lie algebras, but the differences have dramatic consequences. As a result, while their quantum enveloping superalgebras are natural settings to look for more examples of canonical bases, it is not clear that such bases can even exist in general. However, in recent years, I have been working on constructing examples in special cases with my collaborators, and now have several concrete examples to use to develop the general theory.

The projects I plan to pursue in the near future generally fall into the following directions:

- Completing the program of establishing foundational results for covering quantum groups;
- Pursuing a deeper understanding of the fundamental quantum supergroup $\mathrm{U}_q(\mathfrak{gl}(m|n))$, its crystal combinatorics, and the potential for canonical bases;
- Developing further constructions of canonical bases and pursuing a more general theory of canonical bases for (basic) Lie superalgebras;
- Developing HRT categorifications of quantum enveloping superalgebras and their representations.

In the following sections, I will provide a more detailed summary of the background topics; discuss my previous work supporting these directions; and outline the potential projects and outcomes.

1. Background

1.1. **Quantum groups and canonical bases.** The quantum group $\mathrm{U}_q(\mathfrak{g})$ associated to a semisimple (or Kac-Moody) Lie algebra $\mathfrak{g}$ is a deformation of the enveloping algebra depending on a parameter q which recovers the enveloping algebra of the Lie algebra in the limit $q \to 1$. The study of quantum groups associated to Lie algebras, particularly as important examples of non-cocommutative Hopf algebras, was initiated by Drin'feld and Jimbo in 1985. Intense study of the properties of quantum groups followed their introduction, both as objects of physical interest (in the inverse scattering method for quantum mechanical systems) and for their connections to low-dimensional topology and 3-manifold invariants.

One of the most remarkable results was the construction of canonical bases for the quantum group and their integrable modules by Lusztig and Kashiwara ([Lu1, K]) around 1990. The canonical basis of $\mathrm{U}_q(\mathfrak{g})$ is a basis for $\mathrm{U}_q^-(\mathfrak{g})$ (i.e. the subalgebra generated by the negative root vectors) with several remarkable qualities. It is an integral (i.e. $\mathbb{Z}[q, q^{-1}]$) basis invariant under the natural involution $q \mapsto q^{-1}$. The non-zero images of the basis elements on the simple (integrable) modules form a basis of the module, hence the canonical basis is, in this sense, universal. The canonical basis on modules also admits a nice combinatorial description "at $q = 0$" (i.e. in the crystal basis) in terms of Young tableaux, which behaves nicely under tensor products. It is orthonormal (modulo q) with respect to a natural bilinear form. Finally, the transition matrix between the canonical basis and the PBW basis (associated to any order on the roots) is q-unitriangular.

The constructions given by Lusztig and Kashiwara are quite different. Kashiwara's approach was to develop the notion of a crystal: a combinatorial skeleton of a space, together with raising- and lowering-operators encoding the action of the Chevalley generators. These crystals can be explicitly described using Young tableaux [KN] and have many beautiful properties (including a way to interpret branching rules). On the other hand, Lusztig used a geometric construction which realized the half-quantum group as the Grothendieck ring of a category of perverse sheaves. A remarkable consequence of Lusztig's approach is that for simply-laced type, the canonical basis has strong positivity properties.

1.2. **Canonical bases in higher representation theory.** The positivity of canonical bases inspired a number of interesting developments in categorification. Roughly speaking, categorification is the procedure of identifying structures in interesting categories such that applying a suitable forgetful functor decodes (or decategorifies) the structure into a familiar set-based object, like an algebra, or a module. For example, vector spaces categorify the natural numbers, and the decategorification is accomplished by taking the dimension.

In particular, the positivity in the canonical basis strongly suggests that it is the decategorified shadow from some interesting categorification; indeed, in the simply-laced case, the perverse sheaf construction is exactly such a categorification. In particular, from this point of view, the canonical basis essentially provides a Kazhdan-Lusztig theory as follows. The standard basis (i.e. PBW basis) is the shadow of a class of standard objects, whereas the canonical (respectively, dual canonical) basis is a shadow of some nice class, such as tilting or projective, of indecomposable objects (respectively, simple objects). Such a realization allows us to use the transition matrix to compute the decomposition multiplicities of each class in terms of the other. These ideas have been used, for instance, to describe a Kazhdan-Lusztig theory for various Lie superalgebras; see [Br, CLW, BW].

An exciting program of categorification, which was kicked off in the seminal work of Chuang and Rouquier [CR], is the program of "higher representation theory" (henceforth HRT), where the goal is to categorify quantum groups (and other interesting algebras) and their representations as 2-categories (with decategorification accomplished by taking the Grothendieck group of the category). Such categorifications open the way towards constructing categorified knot invariants, specifically knot homologies generalizing Khovanov homology, with the ultimate goal being a way to produce 4-dimensional topological quantum field theories, as predicted by Crane and Frenkel [CF]. Following the groundbreaking work of Khovanov and Lauda [KL] and Rouquier [Rou], much headway has been made on these objectives; see for example [BK, EKL, EQ, VV, Web].

1.3. **Quantum groups for Lie superalgebras.** A natural next step is to generalize these results to quantum groups associated to Lie superalgebras. Several papers in the interim have defined and determined the properties of quantum superalgebras [BKM, Ge, KKT, Ya1]. Of particular interest are the variety of constructions of crystal bases for representations that can be found in the super setting, and the variation in the combinatorial models [BKK, Je, Kw1, Kw3, MZ]. There are also some known examples of categorifications for quantum supergroups and their representations [EL, KKO, Kh, KS, HW].

However, despite this interest, there has been little progress in generalizing the principal results from the non-super setting. This is largely due to some properties of semisimple (or Kac-Moody)

Lie algebras that fail to hold true in the super setting (see [Kac, ChWa] for details). Perhaps the most wide-reaching difference is the existence of isotropic simple roots, which leads to, among other things, more complicated Serre-type relations and non-Weyl-conjugate positive root systems.

While these complications make the study of these algebras more interesting, they also act as obstructions to the usual methods of constructing canonical bases. In particular, the finite-dimensional representations fail to be completely reducible for most simple finite-dimensional Lie superalgebras, which makes the natural generalization of a crystal basis approach untenable in many cases. There is also a lack of suitable geometric analogues for Lie superalgebras, and it seems it will take more significant creative leaps to generalize Lusztig's perverse sheaf construction to this setting.

2. Previous work

2.1. **Covering quantum groups.** In joint work with Hill and Wang, the study of covering quantum groups was initiated in [CHW1] (motivated by the categorification [HW] and the structural results in [CW]). The new ingredients that differentiate covering quantum groups from the usual notion of quantum group are the addition of a parameter π and new Cartan elements $J = \pi^h$.

The parameter π is assumed to satisfy $\pi^2 = 1$, and we use it to abstract signs coming from super phenomena. For example, in the covering algebra setting, we rewrite a "super commutator" as follows:

$$[x, y] = xy - (-1)^{p(x)p(y)}yx \quad \rightsquigarrow \quad xy - \pi^{p(x)p(y)}yx.$$

This parameter π is categorified by the parity change functor. One should interpret π as a parameter linking the super and non-super quantum groups, so that we may specialize $\pi = \pm 1$ to recover the usual half-quantum group or the half-quantum supergroup associated to a super Cartan datum.

In the process of constructing the integral form and canonical basis for the quantum group of $\mathfrak{osp}(1|2)$, we realized that a simple, but crucial, change in the definition of the quantum group allows for a different collection of finite-dimensional highest weight modules to be considered (see [CW]). These different definitions of the quantum group could be unified by adding a new generator J; speaking heuristically, if we use the familiar notation $K = q^h$ where h is the Cartan generator of $\mathfrak{osp}(1|2)$, then $J = \pi^h$. This lead to the following general definition.

A *covering quantum group* associated to an anisotropic super Cartan datum $(I, \cdot)$ (see [CHW1, §1.1]) is the $\mathbb{Q}(q, \pi)$-algebra with generators E_i, F_i, K_i, and J_i satisfying certain relations (see [CHW1, §2.1]). Most of these relations are covering analogues of the relations for quantum supergroups, but the most informative relation is

$$E_iF_j - \pi^{p(i)p(j)}F_jE_i = \delta_{ij}\frac{J_iK_i - K_i^{-1}}{\pi_i q_i - q_i^{-1}}.$$

There is a natural definition of quantum integers for covering quantum groups given by

$$[n] = \frac{(\pi q)^n - q^{-n}}{\pi q - q^{-1}} \in \mathbb{Z}_{\geq 0}[q, q^{-1}, \pi].$$

Then we may define divided powers in terms of these quantum integers, and use these to define an integral form of the covering quantum group. When the Cartan datum is "bar-consistent", there is a bar-involution on the covering quantum group which satisfies

$$\overline{q} = \pi q^{-1}.$$

While unconventional, this bar-involution has a categorical interpretation (see [HW]), fixes the quantum integers, and restricts to an involution on the integral form.

In the papers [CHW1, CHW2, C1, CH], we carefully studied covering quantum groups and their representation theories. The results we obtained are "lifts" of the corresponding classical results in [Lu3, Part I] and [K]; to wit, it is shown that the covering quantum group has a completely reducible representation theory with integrable simple modules $V(\lambda)$ indexed by *all* dominant weights. (By comparison, the quantum supergroups appearing in the literature previously had to restrict the set of dominant weights to satisfy certain evenness conditions unless the quantum group was defined over the field $\mathbb{C}(q)$. This reflects the properties of integrable modules of Kac-Moody superalgebras.) These integrable simple modules, as well as half of the covering quantum group, admit crystal structures which can be globalized (in the sense of Kashiwara). This allows us to prove the following theorem, which is an analogue of the canonical basis results in [Lu1, K].

Theorem 1 ([CHW2])**.** *The covering quantum group* U *associated to an anisotropic bar-compatible super Cartan datum admits a canonical basis* $\mathcal{B}$ *for* U^- *which descends to a basis for the integrable irreducible highest-weight modules* $V(\lambda)$ *for dominant weights* λ*.*

One feature of this construction is that it is a generalization of the construction of canonical bases for quantum groups. In particular, if b is a canonical basis element of the covering quantum group, then setting $\pi = 1$ yields a canonial basis element of the quantum group.

In [C1], I systematically develop the structure of the modified form $\dot{\mathrm{U}}$ of the covering quantum group (in the sense of Lusztig [Lu3, Chapter 23]). This algebra is, in some sense, the limit as $\lambda \to \infty$ of the modules $V(-\lambda) \otimes V(\lambda)$, where $V(-\lambda)$ is a lowest-weight irreducible module. These tensor product modules admit a bar-involution induced by the quasi-$\mathcal{R}$-matrix of U, and by a standard lemma due to Lusztig, they also have a canonical basis induced by the bar involution and the canonical bases on the tensor factors. In the limit, these bases stabilize and induce a basis on the modified form.

Theorem 2 ([C1])**.** *The modified form* $\dot{\mathrm{U}}$ *of the covering quantum group* U *associated to an anisotropic bar-compatible super Cartan datum admits a canonical basis* $\dot{\mathcal{B}}$ *which is compatible with the canonical bases on* $V(-\lambda) \otimes V(\lambda)$*. Moreover,* $\dot{\mathrm{U}}$ *admits a bilinear form for which* $\dot{\mathcal{B}}$ *is signed-almost-orthonormal.*

It is expected that the bilinear form and canonical basis should help determine a categorification for the modified covering quantum group. Indeed, in rank 1, this has been done by Ellis and Lauda [EL], and a higher-rank generalization will be done in a forthcoming paper by Brundan and Ellis; cf. [BE]. These categorifications are conjectured to provide a way to construct (generalizations of) the "odd Khovanov homology" of Ozsváth, Rasmussen, and Szabó [ORS]: a knot homology which also categorifies the Jones polynomial, which agrees with Khovanov homology with $\mathbb{Z}/2\mathbb{Z}$ coefficients, but is a distinct link invariant in general.

One consequence of this conjecture would be that the colored quantum knot polynomials associated to covering quantum $\mathfrak{osp}(1|2n)$ should be "the same" as those associated to $\mathfrak{so}(1+2n)$. In a recent paper [C2], I show that the representations of covering quantum $\mathfrak{osp}(1|2n)$ can be used to generate knot invariants, in the manner of Reshetikhin and Turaev [RT]. This is accomplished by explicitly describing the maps associated to cups, caps, and crossings, and showing that these maps induce a functor from the category of tangles to the category of U-modules.

On the other hand, in earlier joint work with Fan, Li, and Wang [CFLW], a family of maps $\mathfrak{X}$ called twistors were defined on several versions of the covering quantum group. These maps essentially relate the $\pi = \pm 1$ specializations of a complexified version of the covering quantum group. Indeed, in the simplest version, we can view $\mathfrak{X}$ as a linear monomorphism

$$\mathfrak{X} : \mathrm{U}^-|_{\pi=\pm 1} \to \mathbb{Q}(q, \sqrt{-1}) \otimes_{\mathbb{Q}(q)} \mathrm{U}^-|_{\pi=\mp 1}$$

which twists the multiplication on U^- and yet preserves the canonical basis up to a scalar. In [C1], it is shown that this family induces morphisms on the level of simple modules and some tensor products.

In [C2], I expanded on this framework to show that the twistors essentially describes a functor between the $\pi = \pm 1$ module categories, which monoidal modulo a weight-wise twisting operator. In particular, this functor almost commutes with the cups, caps, and crossings, hence translates between the $\pi = -1$ and $\pi = 1$ versions of the tangle operators. This leads to the following result.

Theorem 3 ([C2])**.** *Let $\mathbf{t} \in \mathbb{C}$ such that $\mathbf{t}^2 = -1$, and let K be a knot. Let $J_K^\lambda(q)$ be the covering knot invariant of K colored by the dominant weight λ. Then $J_K^\lambda(q)|_{\pi=-1} = \mathbf{t}^\star J_K^\lambda(q\mathbf{t}^{-1})|_{\pi=1}$ for some integer $\star$ depending on K and λ.*

2.2. **Quantum enveloping superalgebras of basic type.** The basic type Lie superalgebras are those which are closest, in a certain sense, to classical simple Lie algebras. Indeed, the quintessential basic type Lie superalgebras are $\mathfrak{gl}(m|n)$, the prototypical Lie superalgebra of endomorphisms of an $(m|n)$-dimensional superspace; and $\mathfrak{osp}(m|2n)$, the subalgebra of $\mathfrak{gl}(m|2n)$ preserving an (even) super-symmetric bilinear form. (See [Kac] for the classification of simple Lie superalgebras, and [ChWa] for an expanded discussion of these algebras and their properties.)

One defect of the theory of quantum covering groups is that it cannot address any of the Lie superalgebras of basic type other than $\mathfrak{osp}(1|2n)$. The other basic type algebras have several complicating factors in their structures which differentiate them from their non-super counterparts: the existence of isotropic simple roots, the corresponding small size of the Weyl group, a non-semisimple representation theory (in general). In all, these factors make constructing canonical bases a delicate and non-trivial endeavor.

One promising direction is pursued in [CHW3], where we adopt the quantum shuffle algebra approach used by Leclerc [Lec], Rosso [Ro], and others. We embed the quantum supergroup associated to an arbitrary choice of simple system into a q-shuffle superalgebra, and develop the combinatorics of words to construct several distinquished bases, including a monomial basis and a PBW basis, for the quantum supergroup. Using this approach, we demonstrate the following.

Theorem 4 ([CHW3])**.** *Let $\mathfrak{g}$ be the Lie superalgebra $\mathfrak{gl}(m|1)$ or $\mathfrak{osp}(2|2n)$. Let U be the quantum group associated to the standard simple system of $\mathfrak{g}$. Then U^- admits a pseudo-canonical basis which is equal to a PBW basis modulo q, and which is characterized by almost orthogonality under a bilinear form.*

One defect of the shuffle approach is that different PBW bases may produce different canonical bases in general, as a bar-invariant almost-orthogonal basis is only unique up to sign. In some cases, a signed canonical basis is the best one can hope for when in the super setting (since, for instance, any datum with two odd anti-commuting generators would necessarily have a signed basis), but nevertheless there is reason to believe this should not be the case for these examples.

In the case of $\mathfrak{gl}(m|1)$, I verified that the pseudo-canonical basis is in fact canonical. This is done in two different ways. First, I build on the crystal basis theory developed by Benkart, Kang, and Kashiwara [BKK] and Kwon [Kw2] to deduce the existence of a canonical basis of $\mathfrak{gl}(m|1)$ which is compatible with the standard PBW basis. Second, I develop an analogue of Lusztig's braid operators. In the super setting, due to the fact that isotropic roots have no associated reflection in the Weyl group, these are no longer automorphisms of U. Instead, we must replace the Weyl group with a Weyl groupoid obtained by adding in "formal odd reflections" due to Serganova [Ser]. (See also [HY] for a discussion of properties of general Coxeter groupoids.) As a consequence, the odd braid operators instead define isomorphisms between presentations of the quantum group with different choices of generalized Cartan matrices.

Theorem 5 ([C3])**.** *Let* U *be the quantum group associated to the standard simple system of* $\mathfrak{gl}(m|1)$*. The pseudo-canonical basis* $\mathcal{B}$ *of* U^- *is equal to* any *PBW basis modulo* q*, hence is canonical. Moreover,* $\mathcal{B}$ *induces a compatible basis on the highest-weight Kac module* $K(\lambda)$*, and if* λ *is polynomial, a canonical basis on the simple quotient* $V(\lambda)$*. Finally, in the case* $m = 3$*, the canonical basis can be explicitly described by three families of elements. (See* [C3, Theorem 5.2]*.)*

3. Future Directions

Now I will list some of the future directions and projects I plan to pursue in the near future, followed by a brief discussion.

Project 1. *Find a suitable formulation for canonical bases of* $\mathrm{U}_q^-(\mathfrak{gl}(m|n))$ *for* $m, n \geq 2$*.*

The Lie superalgebra $\mathfrak{gl}(m|n)$ is the fundamental example of a Lie superalgebra, and even has an established theory of crystal bases given in [BKK], yet it remains one of the most mysterious cases in terms of constructing canonical bases. This is because the naive generalization of the definition of canonical bases to $\mathfrak{gl}(m|n)$ must fail, largely due to the following fact. Let $\mathrm{U}_q(\mathfrak{gl}(m|n))_{\mathrm{ev}}$ be the subalgebra of $\mathrm{U}_q(\mathfrak{gl}(m|n))$ (defined with respect to the standard simple roots) generated by E_i, F_i, K_i where α_i is an even root. Then we have an isomorphism

$$\mathrm{U}_q(\mathfrak{gl}(m|n))_{\mathrm{ev}} = \mathrm{U}_q(\mathfrak{gl}(m|0)) \otimes \mathrm{U}_q(\mathfrak{gl}(0|n)) \cong \mathrm{U}_q(\mathfrak{gl}(m)) \otimes \mathrm{U}_{q^{-1}}(\mathfrak{gl}(n)).$$

The difference of the quantum parameter being q or q^{-1} (which one might call the *chirality* in q) is not cosmetic and poses a significant obstruction to allowing canonical bases as defined above. Indeed, any reasonable definition of a canonical basis on $\mathrm{U}_q^-(\mathfrak{gl}(m|n))$ should be compatible with the canonical bases of the subalgebras $\mathrm{U}_q^-(\mathfrak{gl}(m))$ and $\mathrm{U}_{q^{-1}}^-(\mathfrak{gl}(n))$. However, this would require the transition matrix from a PBW basis to be chiral: $\mathbb{Z}[q]$-unitriangular on the $\mathfrak{gl}(m)$ part and $\mathbb{Z}[q^{-1}]$-unitriangular on the $\mathfrak{gl}(n)$ part. (See also the definition of Kashiwara operators in [BKK] for an analogue of this phenomenon on the level of crystals.) It is not clear how to interpolate between these two requirements to get a condition on all of $\mathrm{U}_q^-(\mathfrak{gl}(m|n))$, and if we do force both parts to be either $\mathbb{Z}[q]$ or $\mathbb{Z}[q^{-1}]$-triangular, then the basis necessarily changes depending on the choice of PBW basis. (See [C3, Example 2.7] for an explicit example of this phenomenon.)

Despite this setback, there is reason to believe that an analogue of canonical basis exists if we can find the correct conditions. Indeed, let $\mathrm{U}_q^-(\mathfrak{gl}(1|m|1))$ be the half-quantum group associated to the non-standard system of simple roots for $\mathfrak{gl}(m|2)$ which has exactly two disconnected isotropic roots. In other words, this would be the half-quantum group associated to the Dynkin diagram

$$\otimes\!-\!\!-\!\bigcirc\!-\!\!-\ \cdots\ -\!\!-\!\bigcirc\!-\!\!-\!\otimes$$

where the crossed nodes indicate odd isotropic roots, and all the remaining roots are non-isotropic and even. I can show that $\mathrm{U}_q^-(\mathfrak{gl}(1|2|1))$ (and more generally, $\mathrm{U}_q^-(\mathfrak{gl}(1|m|1))$) does have a (signed) canonical basis satisfying the naive definition. (See [C3, Section 5.3] for the $m = 1$ case.) These canonical bases also have crystal structures; see the following figure for a truncated picture of the crystal of $\mathrm{U}_q^-(\mathfrak{gl}(1|2|1))$.

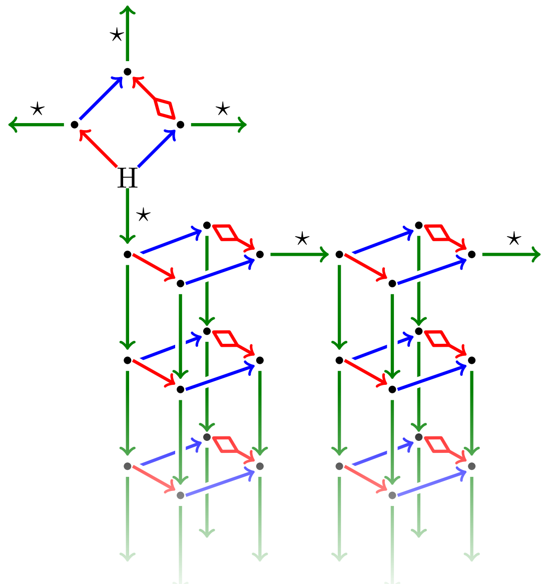

FIGURE 1. This is a conjectural picture of the crystal of $\mathrm{U}_q^-(\mathfrak{gl}(1|2|1))$; the red and blue arrows indicate the isotropic operators, and the diamonds indicate a sign change due to commuting odd operators. The starred arrows indicate some apparent fractal-like repetition of the tower-like structures at the bottom. While conjectural for the whole crystal, this truncated picture has been verified by a Python script.

In this case, the chirality of q essentially vanishes, which allows the naive definition to work. It is not obvious how to use this construction to build a canonical basis for the standard root system case, but it opens the door to apply several other tools. Indeed, we can try to extend this basis to the whole quantum group by taking the modified form $\dot{\mathrm{U}}$. This would give us a more nuanced look at the basis, and potentially use braid operators to concretely produce a basis for the standard half-quantum group. Most importantly, producing a canonical basis for standard $\mathfrak{gl}(2|2)$ would give us insight into the correct definition of canonical bases for Lie superalgebras.

Another way to gain insight on this project is by computing examples. Computation in $\mathrm{U}_q^-(\mathfrak{gl}(2|2))$ can be done by computer, and in fact I have already written some code for computing PBW bases and manipulating them. (Indeed, this is how Figure 1 above was originally produced.) Playing with these computations and looking for patterns could provide valuable insight into the general problem, and would be suitable project for an advanced undergraduate. Not only would it provide motivation to learn about the topics surrounding quantum groups and Lie (super)algebras, but it would also demonstrate the power of programming and experimental computations in mathematical research.

Project 2. *Use the methods from* [C3] *to construct crystal and canonical bases for other basic type Lie superalgebras, and their modules.*

While the methods in [C3] would not lend themselves to a general theory of canonical bases for basic type Lie superalgebras, there are some special cases where a similar strategy would work.

Essentially, the key is to look for simple systems which avoid the chirality phenomenon mentioned in Project 1. For example, the standard root systems of $\mathfrak{osp}(2|2n)$ and $\mathfrak{osp}(2m|2)$ satisfy this constraint, and I have checked that U^- for these algebras do indeed admit canonical bases.

However, novel ideas will be needed for constructing canonical bases of modules in these examples and others. In the ortho-symplectic families, one possible approach would be to use the work of Kwon [Kw3]. In *loc. cit.*, Kwon uses super duality to produce semisimple tensor categories of modules for the quantum groups of ortho-symplectic Lie superalgebras, and demonstrated that these category admit crystal bases. While there are certainly many details to be checked, this could provide a suitable replacement for the results of [BKK] in this setting, and provide a crystal basis approach to the canonical bases.

One interesting loose end of the work in [C3] is that there seems to be a high amount of compatibility between the canonical basis and atypical modules, at least in the low-rank cases. For instance, in the case $m = 2$, the atypical modules do carry a crystal structure after rescaling the isotropic raising operator. For $m = 3$, several families of atypical modules admit a natural lattice $\mathcal{L}(\lambda)$ containing the highest weight vector v_λ and such that $\mathcal{B}(\lambda) = \{b \in \mathcal{B} \mid bv_\lambda \notin q\mathcal{L}(\lambda)\}$ maps bijectively to a basis of $V(\lambda)$.

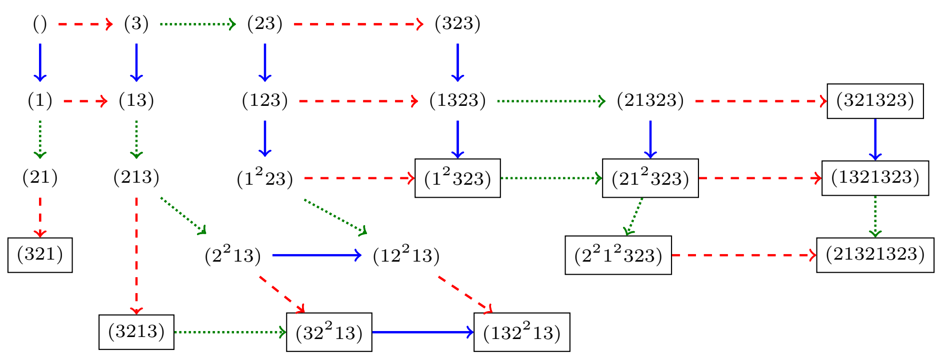

FIGURE 2. The crystal associated to the $\mathrm{U}_q(\mathfrak{gl}(3|1))$-modules $K(\lambda)$, the Kac module of atypical highest weight $\lambda = \epsilon_1 - 3\epsilon_4$. Here, $(i_1^{k_1} i_2^{k_2} \dots i_t^{k_t})$ is shorthand for $F_{i_1}^{(k_1)} \dots F_{i_t}^{(k_t)} 1_\lambda$. The boxed elements are those basis elements which are zero modulo q in the simple quotient $V(\lambda)$

This strongly suggests that even in the atypical case, there is some interesting crystal structure to be determined. However, in general, the Kashiwara operators as defined in [C3] (based on [BKK, Kw2]) are not correct for defining such a crystal, as evidenced in the $m = 2$ case. This leads to two natural questions: can we find a combinatorial model for $\mathfrak{gl}(m|1)$ crystals of atypical weights, and can we realize this model through suitably defined Kashiwara operators to recover the compatibility observed in [C3]?

While these are interesting in their own right, understanding these questions could offer insight into realizing crystals of modules of the quantum groups $\mathfrak{gl}(m|n)$ associated to non-standard choices of simple roots. On the combinatorial level, such crystals were studied by Kwon [Kw1]; however, they are disconnected in general, which makes them significantly more challenging to work with. However, the half-quantum group does admit a canonical basis with a natural connected crystal structure; see for example Figure 1 above. The problem of finding a compatible crystal structure on modules seems related to finding a suitable crystal structure on atypical modules.

One way to gain some traction on this question would be to produce more examples and try to isolate what the underlying combinatorial crystal should be modeled by. This is a project that could be suitable for an advanced undergraduate to play with, and would provide a good motivation for learning more about crystals and their underlying combinatorial models, like Young diagrams and tableaux.

Project 3. *Unify the three constructions of the canonical basis of* $\mathrm{U}_q(\mathfrak{osp}(1|2n))$.

At this point, there are three different ways to construct the canonical bases:

- crystal bases [CHW2];
- through PBW bases using the quantum shuffle algebra [CHW3];
- through PBW bases using braid operators [CH].

It is natural, but not automatic, that these bases should all agree. It should be possible to prove this using analogues of the standard arguments from the non-super case, and in particular would be a suitable project for an advanced undergraduate to be introduced to quantum groups and the ways to construct their canonical bases.

Project 4. *Motivated by the categorification* [KS]*, find similar categorifications for other basic type Lie superalgebras with canonical bases.*

In their paper [KS], Khovanov and Sussan develop a broad theory to use in categorifying one half of standard quantum $\mathfrak{gl}(m|1)$. The key idea is to develop a diagrammatic algebra in the spirit of KLR algebras, but with a dg-algebra structure motivated by an earlier $m = 2$ categorification by Khovanov. These are examples of negative dg gradual algebras, which are certain infinite-dimensional bi-graded dg algebras defined in *loc. cit.* which have a similar representation theory to finite-dimensional algebras.

Given this framework, there are several natural directions in which their work can be extended. One natural extension is to attempt to categorify other half-quantum groups of basic type Lie superalgebras, both for the standard and non-standard choices of root systems. Depending on the precise datum, this would involve modifying the definition of the diagrammatic algebra in *loc. cit.* in such a way that it remains a negative dg gradual algebra (or at least, close enough that it still has similar nice properties). Then similar arguments should allow one to derive the desired categorification result.

Another natural direction is to deduce categorifications of simple modules of $\mathfrak{gl}(m|1)$. For Kac modules and polynomial modules, this should be possible in an entirely analogous way to the non-super case: take appropriate cyclotomic quotients of the diagrammatic algebra. For the remaining atypical modules, it is a more subtle question: in the case λ is atypical, the kernel of the map $\mathrm{U}^- \to V(\lambda)$ generally is not generated by the elements $F_i^{\langle h_i,\lambda\rangle+1}$. This case is somewhat less immediately interesting, however.

Yet another interesting direction is to try to categorify the modified form of the quantum group. This would be a much more non-trivial endeavor, requiring not only to extend the diagrammatics dramatically, as in part 3 of [KL] (see also [EL]), but also for the theory of categorical actions to be well-understood. In the short term, it would be to answer these questions for the small rank cases of $\mathfrak{gl}(1|1)$ and $\mathfrak{gl}(2|1)$. Indeed, in these cases there are already some detailed categorifications using geometry [Tia], topology [EPV], and category $\mathcal{O}$ [Sar].

Project 5. *Study the structure and representation theory of (covering) quantum supergroups at a root of unity.*

An important facet of quantum groups is specialization of the parameter q to a root of unity. For the quantum group associated to a semisimple Lie algebra, Lusztig [Lu2] demonstrated that specialization at a root of unity "approximates" the modular representation theory of the Lie algebra and corresponding algebraic group. Moreover, there is an algebra homomorphism called the quantum Frobenius homomorphism which generalizes (in a suitable sense) the classical Frobenius homomorphism (cf. [Lu3, Chapter 35]).

Given the many parallels already developed between quantum groups and covering quantum groups, we expect that the results in [Lu3, Part V] will have analogues in the covering algebra setting. In particular, I plan to construct an analogue of the quantum Frobenius map and a small covering quantum group. This should provide a bridge between the modular representation theory of Kac-Moody Lie algebras and their super counterparts. This could be particularly interesting in the context of categorifying quantum groups at a root of unity, as in [EQ], as the twistor isomorphisms of [CFLW] would provide a way to twist the specialization of the parameter. Moreover, given the results in [C2], one could ask what, if any, 3-manifold invariants arise from $\mathrm{U}_q(\mathfrak{osp}(1|2n))$ for q a root of unity. Presumably, they would be related to those from $\mathrm{U}_{\mathbf{t}^{-1}q}(\mathfrak{so}(1+2n))$.

In a similar vein, we may study quantum supergroups of basic type at roots of unity as well. We expect that a version of quantum Frobenious maps and small quantum groups would certainly exist in this setting. One parallel with the classical theory that would be interesting to explore is whether the representations of the quantum supergroup at roots of unity approximates modular representations of Lie superalgebras. An important problem in this setting would be formulating an analogue of the Lusztig conjecture, at least for large primes p.

Lusztig's conjecture is a modular version of the famous Kazhdan-Lusztig conjecture, and heuristically these conjectures have the same content: characters of standard modules and simple modules can be expressed in terms of one another with coefficients equal to Kazhdan-Lusztig polynomials evaluated at 1. The crucial observation in this case is that the formal similarity behind quantum groups at roots of unity and modular representation theory can be used to prove Lusztig's conjecture for modular representations by proving the analogous statement for the quantum group, at least when the characteristic is sufficiently large. This pulls the problem back to being a characteristic zero problem, which makes it somewhat more amenable to direct attack.

To take a similar approach for Lie superalgebras, we would first expect there to be a characteristic zero version: that is, a Kazhdan-Lusztig theory for the Lie superalgebra. Such a theory was first formulated by Brundan [Br] for the Lie superalgebra $\mathfrak{gl}(m|n)$. This Brundan-Kazhdan-Lusztig conjecture was proved recently by Cheng, Lam, and Wang [CLW] using Brundan's ideas in concert with dualities for the general linear Lie superalgebra. (Another proof was given by Brundan, Webster, and Losev [BLW] using the framework of tensor product categorifications developed by Webster and Losev.) More recently, Bao and Wang [BW] have used quantum symmetric pairs to extend and prove the BKL conjecture for $\mathfrak{osp}(2m+1|2n)$, which Bao subsequently extended to $\mathfrak{osp}(2m|2n)$ [Ba].

Project 6. *Categorify the tensor product modules of covering quantum* $\mathfrak{sl}_2$ *(and more generally, quantum covering groups) à la Webster.*

One of the more mysterious developments in homological knot invariants was the discovery of odd Khovanov homology due to Ozsváth, Rasmussen, and Szabó [ORS]. This is a knot homology whose Euler characteristic yields the Jones polynomial of the knot, yet is a distinct knot invariant from Khovanov homology. The adjective "odd" comes from the fact that odd Khovanov homology and Khovanov homology agree when taken with $\mathbb{Z}/2\mathbb{Z}$ coefficients.

One would like a conceptual explanation of the existence of this odd Khovanov homology, and, if possible, a machine for producing more examples of "odd knot homologies" categorifying other quantum knot invariants. One way to approach this problem is through HRT. Indeed, just as quantum knot invariants can be constructed through the representation theory of quantum groups via Reshetikhin and Turaev's procedure [RT], Webster [Web] has demonstrated that that this mechanism can be categorified to produce knot homologies through categorified representations of the quantum group. In particular, Webster's work finishes the diagram in Figure 3 below.

To explain odd Khovanov homology, we wish to produce a parallel diagram using a suitable choice of (categorified) quantum groups in the right column. It has been conjectured that Lie superalgebras (and in particular, covering quantum group of $\mathfrak{osp}(1|2)$) would provide the appropriate "odd" analogue. Heuristic evidence for this has been given by Mikhaylov and Witten [MW], and [C2] provides the decategorified link between these quantum covering groups and the usual quantum link invariants of type B. Thus we expect the diagram in Figure 4 to have an analogue of Webster's work filling in the dotted arrow.

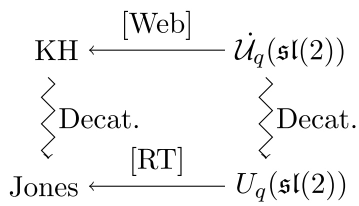

FIGURE 3

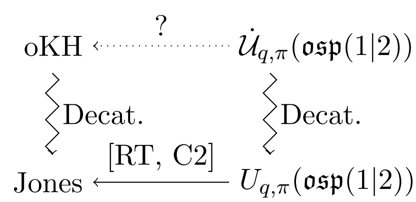

FIGURE 4

In other words, the work that remains is to produce the HRT machinery. There has been much activity in this direction. Indeed, the covering quantum group has been categorified in rank one by Ellis and Lauda [EL], and in general in forthcoming work of Brundan and Ellis [BE]. Moreover, the integrable modules have already been categorified in rank one by [EKL] and the general case by Kang, Kashiwara and Oh [KKO]. However, there are many aspects of the framework of supercategorifications that still need to be developed; in particular, it still remains to find a generalization of the tensor product categorifications of [LW] in the supercategorification setting, and then there is a substantial amount of machinery in [Web] that needs to be generalized to this new setting. As such, this will be a much longer-term project, likely involving collaboration with a number of other authors to complete.

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