Degree-degree correlations in Directed Networks

Pim van der Hoorn
University of Twente, Netherlands
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Introduction

Degree-degree correlations
Directed Configuration Model
Complex Networks
Complex Networks

**Complex Networks**: large (simple) Graphs, with non-trivial structure
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\[ n \geq 10^6 \]
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Examples:
Complex Networks

Complex Networks: large (simple) Graphs, with non-trivial structure

Examples:
- Internet, routers and wires
Complex Networks

Complex Networks: large (simple) Graphs, with non-trivial structure

Examples:

- Internet, routers and wires
- WWW, web-pages and hyperlinks
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Complex Networks: large (simple) Graphs, with non-trivial structure

Examples:
- Internet, routers and wires
- WWW, web-pages and hyperlinks
- Brain, neurons and neural pathways
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- Internet, routers and wires
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- Social networks, people and relations
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- Many more...
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picture from http://www.humanconnectomeproject.org
Degree distributions

\[ p(k) \approx k^{\gamma - 1} \quad \frac{1}{\gamma} \leq \gamma \leq \frac{3}{2} \Rightarrow \mathbb{E}[D] < \infty \]

\[ \mathbb{E}[D^2] = \infty \]
Degree distributions

Loglog plot distribution in-degrees of English Wikipedia

Degree-degree correlations
Degree distributions

Loglog plot distribution in-degrees of English Wikipedia

\[ p(k) \approx k^{-\gamma-1} \]

Degree-degree correlations
Degree distributions

Loglog plot distribution in-degrees of English Wikipedia

\[ p(k) \approx k^{-\gamma - 1} \]

\[ 1 < \gamma \leq 3 \]
Degree distributions

$p(k) \approx k^{-\gamma-1}$

$1 < \gamma \leq 2$
Degree distributions

Loglog plot distribution in-degrees of English Wikipedia

\[ p(k) \approx k^{-(\gamma-1)} \]

\[ 1 < \gamma \leq 2 \quad \Rightarrow \quad \mathbb{E}[D] < \infty \]
Degree distributions

$p(k) \approx k^{-\gamma-1}$

$1 < \gamma \leq 2 \quad \Rightarrow \quad \mathbb{E}[D] < \infty \quad \mathbb{E}[D^2] = \infty$
Introduction

Degree-degree correlations

Directed Configuration Model
Definition & Notations

Index degree type by $\alpha, \beta \in \{+, -\}$.
Definition & Notations

Given a directed graph \( G = (V, E) \).
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\[ i \rightarrow j \]
Definition & Notations

Given a directed graph $G = (V, E)$.

$$i \rightarrow j$$

$\alpha, \beta \in \{+,-\}$
Definition & Notations

Given a directed graph $G = (V, E)$.

$$i \to j$$

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Given a directed graph $G = (V, E)$.

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Four types of degree-degree correlation
Four types of degree-degree correlation

- Out-In
- In-Out
- Out-Out
- In-In
Degree-degree correlations in practice
Degree-degree correlations in practice

- Information flow neural networks.
- Stability of P2P networks under attack.
- Epidemics on networks.
- Network Observability.
- Opinion dynamics based on social influence.
- Collaboration in social networks.
Degree-degree correlations in practice

- Information flow neural networks.
- Stability of P2P networks under attack.
- Epidemics on networks.
- Network Observability.
- Opinion dynamics based on social influence.
- Collaboration in social networks.
- ...
Pearson’s correlation coefficients

Given a set of $m$ joint measurements $\{X_i, Y_i\}_{1 \leq i \leq m}$

\[
r(X, Y) = \frac{\sum_{i=1}^{m} X_i Y_i - \frac{1}{m} \sum_{i=1}^{m} X_i \sum_{i=1}^{m} Y_i}{\sqrt{\text{Var}(X)} \sqrt{\text{Var}(Y)}}
\]

Given a graph $G_n$ of size $n$, pick $\alpha, \beta \in \{+, -\}$.

We have $E$ joint measurements $\{D_{\alpha i}, D_{\beta j}\}_{i \rightarrow j}$

\[
r_{\beta \alpha}(G_n) := r(D_{\alpha}, D_{\beta})
\]

Pearson’s correlation coefficients

Given a set of \( m \) joint measurements \( \{X_i, Y_i\}_{1 \leq i \leq m} \)
Pearson’s correlation coefficients

Given a set of $m$ joint measurements $\{X_i, Y_i\}_{1 \leq i \leq m}$

$$r(X, Y) = \frac{1}{m} \sum_{i=1}^{m} X_i Y_i - \frac{1}{m^2} \sum_{i=1}^{m} X_i \sum_{i=1}^{m} Y_i$$

$$\sqrt{\text{Var}(X)} \sqrt{\text{Var}(Y)}$$
Pearson’s correlation coefficients

Given a set of $m$ joint measurements $\{X_i, Y_i\}_{1 \leq i \leq m}$

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$$\sqrt{\text{Var}(X)} \sqrt{\text{Var}(Y)}$$

$$\text{Var}(X) = \frac{1}{m} \sum_{i=1}^{m} X_i^2 - \frac{1}{m^2} \left( \sum_{i=1}^{m} X_i \right)^2$$
Pearson’s correlation coefficients

Given a set of $m$ joint measurements $\{X_i, Y_i\}_{1 \leq i \leq m}$

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$$\text{Var}(X) = \frac{1}{m} \sum_{i=1}^{m} X_i^2 - \frac{1}{m^2} \left( \sum_{i=1}^{m} X_i \right)^2$$

Given a graph $G_n$ of size $n$, pick $\alpha, \beta \in \{+, -\}$.

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Pearson’s correlation coefficients

Given a set of \( m \) joint measurements \( \{X_i, Y_i\}_{1 \leq i \leq m} \)

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Given a graph \( G_n \) of size \( n \), pick \( \alpha, \beta \in \{+, -\} \).

We have \( E \) joint measurements \( \{D_i^\alpha, D_j^\beta\}_{i \rightarrow j} \)

\[
 r_\beta^n(G_n) := r(D^\alpha, D^\beta)
\]
Pearson’s correlation coefficients

Given a set of \( m \) joint measurements \( \{X_i, Y_i\}_{1 \leq i \leq m} \)

\[
    r(X, Y) = \frac{1}{m} \sum_{i=1}^{m} X_i Y_i - \frac{1}{m^2} \sum_{i=1}^{m} X_i \sum_{i=1}^{m} Y_i \sqrt{\text{Var}(X)} \sqrt{\text{Var}(Y)}
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\]

Given a graph \( G_n \) of size \( n \), pick \( \alpha, \beta \in \{+, -\} \).

We have \( E \) joint measurements \( \{D_i^\alpha, D_j^\beta\}_{i \rightarrow j} \)

\[
    r_\beta^\beta(G_n) := r(D^\alpha, D^\beta)
\]

Consistency in random graphs

Consider a graph $G_n$, of size $n$, sampled from some ensemble $p\alpha(k) = \frac{1}{n} \sum_{i=1}^{n} 1\{D\alpha_i = k\}$ as $n \to \infty$

$p\beta\alpha(k,\ell) = \frac{1}{E} \sum_{i \to j} 1\{D\alpha_i = k, D\beta_j = \ell\} P\to P(D\alpha = k, D\beta = \ell)

\[ r\beta\alpha(G_n) P\to r(D\alpha, D\beta) = \text{Cov}(D\alpha, D\beta) \sqrt{\text{Var}(D\alpha) \text{Var}(D\beta)} \]

$P\alpha(k) \approx k - \gamma\alpha - 1 1 < \gamma\alpha \leq 2$

$E[D\alpha] < \infty$

$E[(D\alpha)^2] = \infty$
Consistency in random graphs

Consider a graph $G_n$, of size $n$
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$$p^\alpha(k) := \frac{1}{n} \sum_{i=1}^{n} \mathbb{1}\{D_i^\alpha = k\}$$
Consistency in random graphs

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$$p^\alpha(k) := \frac{1}{n} \sum_{i=1}^{n} \mathbb{1}\{D_i^\alpha = k\}$$

$$p^\beta(k, \ell) := \frac{1}{E} \sum_{i \to j} \mathbb{1}\{D_i^\alpha = k, D_j^\beta = \ell\}$$
Consistency in random graphs

Consider a graph $G_n$, of size $n$, sampled from some ensemble

$$p^\alpha(k) := \frac{1}{n} \sum_{i=1}^{n} \mathbb{1}\{D_i^\alpha = k\}$$

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Consider a graph $G_n$, of size $n$, sampled from some ensemble

$$p^\alpha(k) := \frac{1}{n} \sum_{i=1}^{n} \mathbb{1}\{D_i^\alpha = k\} \xrightarrow{P} P^\alpha(k) \quad \text{as } n \to \infty$$

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$$p^\beta_\alpha(k, \ell) := \frac{1}{E} \sum_{i \to j} \mathbb{1}\{D^\alpha_i = k, D^\beta_j = \ell\} \xrightarrow{P} \mathbb{P}\left(D^\alpha = k, D^\beta = \ell\right)$$
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$$r^\beta_{\alpha}(G_n) \xrightarrow{\mathbb{P}} r(D^\alpha, D^\beta) = \frac{\text{Cov}(D^\alpha, D^\beta)}{\sqrt{\text{Var}(D^\alpha)\text{Var}(D^\beta)}}$$
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$$P^\alpha(k) \approx k^{-\gamma^\alpha - 1} \quad 1 < \gamma^\alpha \leq 2$$
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$$r^\beta(G_n) \xrightarrow{\mathbb{P}} r(D^\alpha, D^\beta) = \frac{\text{Cov}(D^\alpha, D^\beta)}{\sqrt{\text{Var}(D^\alpha)\text{Var}(D^\beta)}}$$

$$P^\alpha(k) \approx k^{-\gamma^\alpha - 1} \quad 1 < \gamma^\alpha \leq 2 \quad \mathbb{E}[D^\alpha] < \infty$$
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Consider a graph $G_n$, of size $n$, sampled from some ensemble

$$p^\alpha(k) := \frac{1}{n} \sum_{i=1}^{n} \mathbb{1}\{D^\alpha_i = k\} \xrightarrow{\mathbb{P}} P^\alpha(k) \quad \text{as } n \to \infty$$

$$p^\beta(k, \ell) := \frac{1}{E} \sum_{i \to j} \mathbb{1}\{D^\alpha_i = k, D^\beta_j = \ell\} \xrightarrow{\mathbb{P}} \mathbb{P}\left(D^\alpha = k, D^\beta = \ell\right)$$

$$r^\beta(\mathbb{G}_n) \xrightarrow{\mathbb{P}} r(D^\alpha, D^\beta) = \frac{\text{Cov}(D^\alpha, D^\beta)}{\sqrt{\text{Var}(D^\alpha)\text{Var}(D^\beta)}}$$

$$P^\alpha(k) \approx k^{-\gamma^\alpha - 1} \quad 1 < \gamma^\alpha \leq 2 \quad \mathbb{E}[D^\alpha] < \infty \quad \mathbb{E}\left[(D^\alpha)^2\right] = \infty$$
Convergence of Pearson’s correlation coefficients

Theorem 1 (vdH and Litvak, 2014)

Let $\alpha, \beta \in \{+,-\}$. Then there exists an area $A^{\beta,\alpha} \subset \mathbb{R}^2$ such that if $\{G_n\}_{n \in \mathbb{N}}$ is a sequence of graphs with scale-free degree distributions where the tail-exponents $(\gamma^+, \gamma^-) \in A^{\beta,\alpha}$, then

$$\lim_{n \to \infty} r^{\beta,\alpha}(G_n) \geq 0,$$

for all $\alpha, \beta \in \{+,-\}$.
Convergence of Pearson’s correlation coefficients

Theorem 1 (vdH and Litvak, 2014)

Let \( \alpha, \beta \in \{+, -\} \). Then there exists an area \( A_{\beta \alpha} \subset \mathbb{R}^2 \) such that if \( \{G_n\}_{n \in \mathbb{N}} \) is a sequence of graphs with scale-free degree distributions where the tail-exponents \( (\gamma^+, \gamma^-) \in A_{\beta \alpha} \),

\[
\lim_{n \to \infty} r_{\beta \alpha}(G_n) \geq 0,
\]

for all \( \alpha, \beta \in \{+, -\} \).

[Litvak and van der Hofstad, Phys Rev E, 2013]
Convergence of Pearson’s correlation coefficients

Theorem 1 (vdH and Litvak, 2014)

Let $\alpha, \beta \in \{+, -\}$. Then there exists an area $A^\beta_\alpha \subset \mathbb{R}^2$ such that if \( \{G_n\}_{n \in \mathbb{N}} \) is a sequence of graphs with scale-free degree distributions where the tail-exponents $(\gamma_+, \gamma_-) \in A^\beta_\alpha$,

$$\lim_{n \to \infty} r^\beta_\alpha(G_n) \geq 0.$$
Convergence of Pearson’s correlation coefficients

**Theorem 1 (vdH and Litvak, 2014)**

Let $\alpha, \beta \in \{+, -\}$. Then there exists an area $A^\beta_\alpha \subset \mathbb{R}^2$ such that if $\{G_n\}_{n \in \mathbb{N}}$ is a sequence of graphs with scale-free degree distributions where the tail-exponents $(\gamma_+, \gamma_-) \in A^\beta_\alpha$,

$$\lim_{n \to \infty} r^\beta_\alpha(G_n) \geq 0.$$ 

$$1 < \gamma_\pm \leq 2 \in A^\beta_\alpha, \text{ for all } \alpha, \beta \in \{+, -\}$$

[Litvak and van der Hofstad, Phys Rev E, 2013]
Ranking the observations (Spearman’s rho)

Given a graph $G_n$ of size $n$, $\alpha, \beta \in \{+, -\}$

Rank the degrees in descending order

We have $E$ joint measurements $\{D_\alpha^i, D_\beta^j\}$ $i \rightarrow j \Rightarrow \{R_\alpha^i, R_\beta^j\}$ $i \rightarrow j$

Compute Pearson's correlation coefficient on $\rho_{\beta \alpha}(G_n) := r(R_\alpha, R_\beta)$
Ranking the observations (Spearman’s rho)

Given a graph $G_n$ of size $n$, $\alpha, \beta \in \{+, -\}$
Ranking the observations (Spearman’s rho)

Given a graph $G_n$ of size $n$, $\alpha, \beta \in \{+, -\}$

We have $E$ joint measurements $\{D_i^\alpha, D_j^\beta\}_{i \rightarrow j}$
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Rank the degrees in descending order

We have $E$ joint measurements $\{D_i^\alpha, D_j^\beta\}_{i \rightarrow j} \Rightarrow \{R_i^\alpha, R_j^\beta\}_{i \rightarrow j}$

Compute Pearson's correlation coefficient on $\{R_i^\alpha, R_j^\beta\}_{i \rightarrow j}$
Ranking the observations (Spearman’s rho)

Given a graph \( G_n \) of size \( n \), \( \alpha, \beta \in \{+, -\} \)

Rank the degrees in descending order

We have \( E \) joint measurements \( \{D_i^\alpha, D_j^\beta\}_{i \rightarrow j} \Rightarrow \{R_i^\alpha, R_j^\beta\}_{i \rightarrow j} \)

Compute Pearson’s correlation coefficient on \( \{R_i^\alpha, R_j^\beta\}_{i \rightarrow j} \)

\[
\rho_\alpha^\beta(G_n) := r(R_i^\alpha, R_j^\beta)
\]
Ranking the observations (Spearman’s rho)

Given a graph $G_n$ of size $n$, $\alpha, \beta \in \{+, -\}$

Rank the degrees in descending order

We have $E$ joint measurements $\{D_i^\alpha, D_j^\beta\}_{i \rightarrow j} \Rightarrow \{R_i^\alpha, R_j^\beta\}_{i \rightarrow j}$

Compute Pearson's correlation coefficient on $\{R_i^\alpha, R_j^\beta\}_{i \rightarrow j}$

$$\rho^\beta_{\alpha}(G_n) := r(R^\alpha, R^\beta)$$
Theorem 2 (vdH and Litvak, 2014)

Let $\{G_n\}_{n \in \mathbb{N}}$ be a sequence of random graphs, $\alpha, \beta \in \{+, -\}$ and suppose there exist integer valued random variables $D^\alpha$ and $D^\beta$ such that

$$p^\beta_{\alpha}(k, \ell) \xrightarrow{\mathbb{P}} \mathbb{P}\left(D^\alpha = k, D^\beta = \ell\right) \text{ as } n \to \infty.$$ 

Then, as $n \to \infty$,

$$p^\beta_{\alpha}(G_n) \xrightarrow{\mathbb{P}} \rho\left(D^\alpha, D^\beta\right)$$
Statistical consistency Spearman’s rho

Theorem 2 (vdH and Litvak, 2014)

Let $\{G_n\}_{n \in \mathbb{N}}$ be a sequence of random graphs, $\alpha, \beta \in \{+, -\}$ and suppose there exist integer valued random variables $D^\alpha$ and $D^\beta$ such that

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Theorem 2 (vdH and Litvak, 2014)

Let \( \{G_n\}_{n \in \mathbb{N}} \) be a sequence of random graphs, \( \alpha, \beta \in \{+, -\} \) and suppose there exist integer valued random variables \( D^\alpha \) and \( D^\beta \) such that

\[
p^\beta_\alpha(k, \ell) \xrightarrow{\mathbb{P}} \mathbb{P}\left(D^\alpha = k, D^\beta = \ell\right) \quad \text{as } n \to \infty.
\]

Then, as \( n \to \infty \),

\[
p^\beta_\alpha(G_n) \xrightarrow{\mathbb{P}} \rho\left(D^\alpha, D^\beta\right)
\]

Introduction
Degree-degree correlations
Directed Configuration Model
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Directed Configuration Model

[Chen and Olvera-Cravioto, Stoch. Syst., 2013]
Directed Configuration Model

\[ F^+ \]

\[ v_1 \]

\[ v_2 \]

\[ \vdots \]

\[ v_n \]

[Chen and Olvera-Cravioto, Stoch. Syst., 2013]
Directed Configuration Model

$F^+$

$F^-$

$v_1$

$F^+$

$F^-$

$v_2$

$\ldots$

$v_n$

[Chen and Olvera-Cravioto, Stoch. Syst., 2013]
Directed Configuration Model

$F^+$

$v_1$

\[ \rightarrow \]

$v_2$

\[ \rightarrow \]

\[ \vdots \]

$v_n$

\[ \rightarrow \]

$F^-$

$v_1$

\[ \rightarrow \]

$v_2$

\[ \rightarrow \]

\[ \vdots \]

$v_n$

\[ \rightarrow \]

[Chen and Olvera-Cravioto, Stoch. Syst., 2013]
Directed Configuration Model

\[ F^+ \]

\[ \vdots \]

\[ F^- \]

[Chen and Olvera-Cravioto, Stoch. Syst., 2013]
Directed Configuration Model

General Model

\[ F^+ \]

\[ v_1 \quad v_2 \quad \ldots \quad v_n \]

\[ F^- \]

\[ v_1 \quad v_2 \quad \ldots \quad v_n \]

[Chen and Olvera-Cravioto, Stoch. Syst., 2013]
Directed Configuration Model

Degree-degree correlations

[Chen and Olvera-Cravioto, Stoch. Syst., 2013]

[Chen and Olvera-Cravioto, Stoch. Syst., 2013]
Directed Configuration Model

\[ F^+ \]
\[ v_1 \]
\[ v_2 \]
\[ \vdots \]
\[ v_n \]

\[ F^- \]
\[ v_1 \]
\[ v_2 \]
\[ \vdots \]
\[ v_n \]

[Chen and Olvera-Cravioto, Stoch. Syst., 2013]
Directed Configuration Model

Repeated Model

$F^+$

$\ldots$

$F^-$

$F^+$

$F^-$

[Chen and Olvera-Cravioto, Stoch. Syst., 2013]
Directed Configuration Model

Degree-degree correlations

[Chen and Olvera-Cravioto, Stoch. Syst., 2013]
Directed Configuration Model

Erased Model

\[ F^+ \]

\[ v_1 \rightarrow v_2 \rightarrow \ldots \rightarrow v_n \]

\[ F^- \]

\[ v_1 \rightarrow v_2 \rightarrow \ldots \rightarrow v_n \]

[Chen and Olvera-Cravioto, Stoch. Syst., 2013]

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Theorem 3 (vdH and Litvak, 2014)

Let $\{G_n\}_{n \in \mathbb{N}}$ be a sequence of graphs of size $n$, generated by either the Repeated or Erased Configuration Model and $\alpha, \beta \in \{+, -\}$.

Then, as $n \to \infty$, $\rho^\beta_\alpha(G_n) \to 0$.
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\]
Spearman’s rho in the Configuration Model

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Use Theorem 2

\[
\rho_{\alpha}^{\beta}(k, \ell) \xrightarrow{\mathbb{P}} \mathbb{P} \left( D^\alpha = k, D^\beta = \ell \right)
\]
Spearman’s rho in the Configuration Model

Theorem 3 (vdH and Litvak, 2014)

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Use Theorem 2

\[
\rho^\beta_\alpha(k, \ell) \xrightarrow{\mathbb{P}} \mathbb{P}(D^\alpha = k, D^\beta = \ell) = \mathbb{P}(D^\alpha = k) \mathbb{P}(D^\beta = \ell)
\]
Erased model in practice
Erased model in practice

Figure: Empirical cdf of $\rho^\beta_\alpha(G_n)$ for ECM graphs with $\gamma_\pm = 2.1$
Erased model in practice

Figure: Empirical cdf of $\rho^\beta(\mathcal{G}_n)$ for ECM graphs with $\gamma_\pm = 1.5$
Why is Out-In different?

\[
\rho - G_n = 0
\]

Degree-degree correlations 18/28
Why is Out-In different?

\[ \rho^- (G_n) = \]

\[ D^+ \quad 0 \quad 1 \]

\[ D^- \]

Degree-degree correlations
Why is Out-In different?

\[ \rho^- (G_n) = 0 \]
Why is Out-In different?

\[ \rho_+^-(G_n) = 0 \]
Why is Out-In different?

\[ \rho^{-}(G_{n}) = 0 \]
Why is Out-In different?

\[ \rho^-(G_n) = 0 \]
Why is Out-In different?

\[ \rho_+^{-1}(G_n) < 0 \]
What about In-Out?

\[ D^+ \quad \quad \quad \quad D^- \]

\[ -1 \quad 0 \quad 1 \]
What about In-Out?

\[ \rho^+_{-}(G_n) = 0 \]
What about In-Out?

\[
\rho^+_-(G_n) = 0
\]
What about In-Out?

\[ \rho^+_\pm(G_n) = 0 \]
What about In-Out?

\[ \rho^+ (G_n) = 0 \]
What about In-Out?

\[ \rho_+^{+}(G_n) = 0 \]
What about In-Out?

\[ \rho_+^-(G_n) = 0 \]
Scaling of $\rho_{\alpha}^\beta$

Let $G_n$ be a graph of size $n$, generated by the ECM and denote by $G^*_{n}$ the graph before the removal of edges. Let $E_{c_{ij}}$ denote the number of erased edges between $i$ and $j$ in ECM.

$$D_{+}'_i = D_{+}_i - n\sum_{j=1}^{n}E_{c_{ij}}.$$
Scaling of $\rho_\beta^\alpha$

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Scaling of $\rho_\alpha^\beta$

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\[
D_{i}^{+'} = D_{i}^{+} - \sum_{j=1}^{n} E_{ij}^c.
\]
Scaling of $\rho_\beta^\alpha$

Let $G_n$ be a graph of size $n$, generated by the ECM and denote by $G_n^*$ the graph before the removal of edges.

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$$D_i^{+'} = D_i^+ - \sum_{j=1}^{n} E_{ij}^c.$$

$$|\rho_+^-(G_n) - \rho_+^-(G_n^*)| = O \left( \frac{1}{E_n} \sum_{i,j=1}^{n} \mathbb{E}_n \left[ E_{ij}^c \right] \right)$$
Scaling of $\rho^\beta_\alpha$

Let $G_n$ be a graph of size $n$, generated by the ECM and denote by $G_n^*$ the graph before the removal of edges.

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$$D_i^{+\prime} = D_i^+ - \sum_{j=1}^{n} E_{ij}^c.$$  

$$\left| \rho_+^- (G_n) - \rho_+^- (G_n^*) \right| = O \left( \frac{1}{E} \sum_{i,j=1}^{n} \mathbb{E}_n [ E_{ij}^c ] \right)$$
A first upper bound
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\[ \sum_{i,j=1}^{n} E_{ij}^{c} \]
A first upper bound

\[ \sum_{i,j=1}^{n} E_{ij}^c = \sum_{i,j=1}^{n} M_{ij} + \sum_{i=1}^{n} S_{ii} \]
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\mathbb{E}_n [S_{ii}] = \frac{D_i^+ D_i^-}{E}
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\[ \frac{1}{E} \sum_{i,j=1}^{n} \mathbb{E}_n [E_{ij}^c] \leq \sum_{i,j=1}^{n} \frac{(D_i^+)^2 (D_j^-)^2}{E^3} + \sum_{i=1}^{n} \frac{D_i^+ D_i^-}{E^2} \]
A first upper bound

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\[ \frac{1}{E} \sum_{i,j=1}^{n} \mathbb{E}_n [E_{ij}^c] \leq \sum_{i,j=1}^{n} \frac{(D_i^+)^2 (D_j^-)^2}{E^3} + O(n^{-1}) \]
A first upper bound

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\[ \frac{1}{E} \sum_{i,j=1}^{n} \mathbb{E}_n [E_{ij}^c] \leq O \left( n^{\frac{2}{\gamma_+} + \frac{2}{\gamma_-} - 3} \right) + O \left( n^{-1} \right) \]
A first upper bound

\[
\sum_{i,j=1}^{n} E_{ij}^c = \sum_{i,j=1}^{n} M_{ij} + \sum_{i=1}^{n} S_{ii}
\]

\[
\mathbb{E}_n [S_{ii}] = \frac{D_i^+ D_i^-}{E} \quad \mathbb{E}_n [M_{ij}] \leq \frac{(D_i^+)^2 (D_j^-)^2}{E^2}
\]

\[
\frac{1}{E} \sum_{i,j=1}^{n} \mathbb{E}_n [E_{ij}^c] \leq O \left( n^{\frac{2}{\gamma_+} + \frac{2}{\gamma_-} - 3} \right)
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\[ \frac{1}{E} \sum_{i,j=1}^{n} \mathbb{E}_n [E_{ij}^c] \leq O \left( n \frac{2}{\gamma_+} + \frac{2}{\gamma_-} - 3 \right) \]
A second upper bound
A second upper bound

\[
\frac{1}{E} \sum_{i,j=1}^{n} \mathbb{E}_n [E_{ij}^c] \leq 1 - \frac{n^2}{E} + \frac{1}{E} \sum_{i,j=1}^{n} \exp \left\{ \frac{D_i^+ D_j^-}{E} \right\}
\]
A second upper bound

\[
\frac{1}{E} \sum_{i,j=1}^{n} \mathbb{E}_n \left[ E_{ij}^c \right] \leq \frac{n^2}{E} \left( \frac{1}{n^2} \sum_{i,j=1}^{n} \frac{D_i^+ D_j^-}{E} - 1 + \frac{1}{n^2} \sum_{i,j=1}^{n} \exp \left\{ \frac{D_i^+ D_j^-}{E} \right\} \right)
\]
A second upper bound

\[
\frac{1}{E} \sum_{i,j=1}^{n} \mathbb{E}_n \left[ E_{ij}^c \right] \leq \frac{n^2}{E} \left( \frac{1}{n^2} \sum_{i,j=1}^{n} \frac{D_i^+ D_j^-}{E} - 1 + \frac{1}{n^2} \sum_{i,j=1}^{n} \exp \left\{ \frac{D_i^+ D_j^-}{E} \right\} \right)
\]
A second upper bound

\[
\frac{1}{E} \sum_{i,j=1}^{n} \mathbb{E}_n \left[ E_{ij}^c \right] \leq O \left( n^{\frac{1}{\gamma^+ \wedge \gamma^-} - 1} \right) + O \left( n^{(\gamma^+ \wedge \gamma^-) - 1} \right)
\]
A second upper bound

\[
\frac{1}{E} \sum_{i,j=1}^{n} \mathbb{E}_n [E_{ij}^c] \leq O \left( n^{\frac{1}{\gamma_+ \wedge \gamma_-} - 1} \right) + O \left( n^{(\gamma_+ \wedge \gamma_-) - 1} \right)
\]

\[
1 < \gamma_+ \leq 2
\]
A second upper bound

\[
\frac{1}{E} \sum_{i,j=1}^{n} \mathbb{E}_{n} [E_{ij}^{c}] \leq O \left( n^{\frac{1}{\gamma_+ \wedge \gamma_-} - 1} \right)
\]

\[1 < \gamma_\pm \leq 2\]
Phase transitions for \( \frac{1}{E} \sum_{i,j=1}^{n} \mathbb{E}_n \left[ E_{ij}^c \right] \)
Phase transitions for $\frac{1}{E} \sum_{i,j=1}^{n} \mathbb{E}_n [E_{ij}^c]$
Phase transitions for $\frac{1}{E} \sum_{i,j=1}^{n} \mathbb{E}_n [E_{ij}]$

\[
\frac{1}{\gamma_+ \wedge \gamma_-} - 1 \leq \frac{2}{\gamma_+} + \frac{2}{\gamma_-} - 3
\]

\[
\frac{1}{E} \sum_{i,j=1}^{n} \mathbb{E}_n [E_{ij}] \leq O \left( n^{\frac{2}{\gamma_+} + \frac{2}{\gamma_-} - 3} \right), \quad O \left( n^{\frac{1}{\gamma_+ \wedge \gamma_-} - 1} \right)
\]
Phase transitions for \( \frac{1}{E} \sum_{i,j=1}^{n} \mathbb{E}_n [E_{ij}^c] \)

\[
\frac{1}{\gamma_+ \wedge \gamma_-} - 1 > \frac{2}{\gamma_+} + \frac{2}{\gamma_-} - 3
\]

\[
\frac{1}{E} \sum_{i,j=1}^{n} \mathbb{E}_n [E_{ij}^c] \leq O\left(n \frac{2}{\gamma_+} + \frac{2}{\gamma_-} - 3\right), \quad O\left(n \frac{1}{\gamma_+ \wedge \gamma_-} - 1\right)
\]
Phase transitions for $\rho^-_+(G_n)$
Phase transitions for $\rho_+(G_n)$

$$
\rho_+(G_n) = O \left( \frac{1}{E} \sum_{i,j=1}^{n} \mathbb{E}_n \left[ E_{ij}^c \right] \right)
$$
Phase transitions for $\rho_+^{-}(G_n)$

$$
\rho_+^{-}(G_n) = O \left( \frac{1}{E} \sum_{i,j=1}^{n} \mathbb{E}_n \left[ E_{ij}^c \right] \right) + O \left( \rho_+^{-}(G_n^*) \right)
$$

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Phase transitions for $\rho_+^-(G_n)$

\[ \rho_+^-(G_n) = O \left( \frac{1}{E} \sum_{i,j=1}^{n} \mathbb{E}_n [ E_{ij}^c ] \right) + O \left( \rho_+^-(G_n^*) \right) \]
Phase transitions for $\rho_+^-(G_n)$

$$\rho_+^-(G_n) = O \left( \frac{1}{E} \sum_{i,j=1}^{n} \mathbb{E}_n [E_{ij}^c] \right) + O \left( \rho_+^-(G_n^*) \right)$$
Phase transitions for $\rho_+(G_n)$

\[
\rho_+(G_n) = O \left( \frac{1}{E} \sum_{i,j=1}^{n} \mathbb{E}_n [E_{ij}^c] \right) + O \left( n^{-1/2} \right)
\]
Phase transitions for $\rho_+^{-}(G_n)$

\[
\frac{1}{\gamma_+ \land \gamma_-} - 1 > -\frac{1}{2}
\]

$\rho_+^{-}(G_n) = O \left( \frac{1}{E} \sum_{i,j=1}^{n} \mathbb{E}_n [E_{ij}^c] \right) + O \left( n^{-1/2} \right)$
Phase transitions for $\rho_+(G_n)$

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Phase transitions for $\rho^-(G_n)$

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\[
\rho^-(G_n) = O \left( \frac{1}{E} \sum_{i,j=1}^{n} E_n [E_{ij}^c] \right) + O \left( n^{-1/2} \right)
\]
Scaling of $\rho_\pm(G_n)$ in practice
Scaling of $\rho^\pm(G_n)$ in practice
Scaling of $\rho_-(G_n)$ in practice

\[
\frac{\rho_-(G_n) - \mathbb{E}[\rho_-(G_n)]}{\mathcal{N}^f(\gamma_+, \gamma_-)}
\]
Scaling of $\rho_+^-(G_n)$ in practice

$$\frac{\rho_+^-(G_n) - \mathbb{E}[\rho_+^-(G_n)]}{N^f(\gamma_+ , \gamma_-)}$$

Degree-degree correlations

$N^{-1+1/(\gamma_+ \wedge \gamma_-)}$ : $N^{(2/\gamma_+)+(2/\gamma_-)-3}$ : $N^{-1/2}$
Scaling of $\rho_+(G_n)$ in practice

\[
\frac{\rho_+(G_n) - \mathbb{E}[\rho_+(G_n)]}{\mathcal{N}^f(\gamma_+, \gamma_-)}
\]

Degree-degree correlations

\[ N^{-1+1/(\gamma_+ \wedge \gamma_-)} \]
\[ N^{(2/\gamma_+)+(2/\gamma_-)-3} \]
\[ N^{-1/2} \]
Scaling of $\rho_+^{-}(G_n)$ in practice

$\rho_+^{-}(G_n) - \mathbb{E}[\rho_+^{-}(G_n)]$

$\mathcal{N}^f(\gamma_+, \gamma_-)$

Degree-degree correlations

$\mathcal{N}^{-1+1/(\gamma_+ \wedge \gamma_-)}$

$\mathcal{N}^{(2/\gamma_+) + (2/\gamma_-)-3}$

$\mathcal{N}^{-1/2}$
Scaling of $\rho^\pm(G_n)$ in practice
Scaling of $\rho^\pm(G_n)$ in practice

\[
\frac{\rho^\pm(G_n) - \mathbb{E}[\rho^\pm(G_n)]}{\mathcal{N}^f(\gamma_+, \gamma_-)}
\]

Degree-degree correlations
Scaling of $\rho_\pm^+(G_n)$ in practice

$$\frac{\rho_\pm^+(G_n) - \mathbb{E}[\rho_\pm^+(G_n)]}{\mathcal{N}^f(\gamma_+, \gamma_-)}$$
Scaling of $\rho_+^-(G_n)$ in practice

$$\frac{\rho_+^-(G_n) - \mathbb{E}[\rho_+^-(G_n)]}{\mathcal{N}^f(\gamma_+^0, \gamma_-^0)}$$
Scaling of $\rho_{-}^{\pm}(G_n)$ in practice

\[
\frac{\rho_{-}^{\pm}(G_n) - \mathbb{E}[\rho_{-}^{\pm}(G_n)]}{\mathcal{N}^{f(\gamma_+, \gamma_-)}}
\]

Degree-degree correlations
Scaling of $\rho_-(G_n)$ in practice

$$\frac{\rho_-(G_n) - \mathbb{E}[\rho_-(G_n)]}{N^{f(\gamma_+, \gamma_-)}}$$

Degree-degree correlations

$N^{-1+1/(\gamma_+ \wedge \gamma_-)}$ : $N^{(2/\gamma_+)+(2/\gamma_-)-3}$ : $N^{-1/2}$
Summary

Degree-degree correlations capture important topological properties of networks.

Pearson’s correlation coefficients are inconsistent for most real-world networks.

Rank correlations (Spearman’s rho) are consistent for a large range of networks.

Directed configuration model is a null model for degree-degree correlations.

Making graphs simple gives rise to structural negative Out-In correlations.

Established and showed three different scaling regimes for $\rho^{-\rho^+}$.
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- Making graphs simple gives rise to structural negative Out-In correlations.
- Established and showed three different scaling regimes for $\rho_{\pm}$. 
