HOMOTOPIE CATEGORIES OF RINGS

Properties and consequences in module categories

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INTRODUCTION

\( R = \) Noncommutative ring with unit

\( \text{Mod} - \text{R} = \) The category of right modules.

\( \text{C}(R) = \) Cochain complexes with chain maps.

\[
\begin{array}{c}
\rightarrow \\
\cdots \\
\rightarrow \quad \overset{d^{n-1}}{X} \\
\rightarrow \quad \overset{d^{n}}{X} \\
\rightarrow \quad \overset{d^{n+1}}{X} \\
\end{array}
\]

\( A = \) Additive subcategory of \( \text{Mod} - \text{R} \)

\( K(A) = \) Homotopy category

- Objects: Cochain complexes.
- Morphisms: Homotopy equivalence classes of cochain maps.
Given \( X, Y \in C(R) \) and \( f : g : X \rightarrow Y \)

\[ f \circ g \iff \exists s^n : X^n \rightarrow Y^{n-1} \text{ with } \]
\[ f^n - s^n = d^y s^n + s^{n+1} d_x \]
Properties of $K(A)$

1. $K(A)$ is a triangulated category
   a) Additive
   b) There is an isomorphism (suspension) $\Sigma: K(A) \to K(A)$
   c) There is a class of composable morphisms
      
      $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma A$
      
      called triangles
Main Idea of the Talk

a) Study $K(A)$ using module theory

$\text{Mod-}R \quad \Rightarrow \quad K(A),\ A \in \text{Mod-}R$

b) Study $K(A)$ using triangulated cat. and obtaining consequences in $\text{Mod-}R$
Theorem 1.1. The homotopy category $K(R$-Proj$)$ is always $\aleph_1$-compactly generated, and as a consequence satisfies Brown representability. But it need not be compactly generated. Precisely

(i) If $R$ is right coherent then $K(R$-Proj$)$ is compactly generated.
(ii) We give an example of an $R$ for which $K(R$-Proj$)$ is not compactly generated.

$\aleph_1$-compactly generated

$K(R$-Proj$)$ has a set of generators $S'$ such that every $S \subseteq S'$ is $\aleph_1$-compact:

For any $f: S \to \bigoplus_{i \in I} K_i$, if $J \subseteq I$ with $|J| < \aleph_1$
From $\text{Mod-R}$ to $K(A)$


If $A$ is a deconstructible class of modules, then $K(A)$ is coreflective in $K(\text{Mod-R})$ ($K(A) \subset K(\text{Mod-R})$ has a right adjoint).

$A$ is deconstructible if there exists a set of objects $\mathcal{S} \subseteq A$ such that $\forall A \in A$

- $A_\alpha \subseteq A_{\alpha+1}$
- $A_\beta = \bigcup_{\alpha \leq \beta} A_\alpha$, $\beta$ limit
- $A_{\alpha+1} / A_\alpha \in \mathcal{S}$

Then $(A_\alpha | \alpha < \kappa)$ is the $\mathcal{S}$-filtration of $A$
Example 1: Finite length modules

$M$ has finite length if it contains a chain of submodules

$$0 \leq M_1 \leq M_2 \leq M_3 \leq \cdots \leq M_n = M$$

such that there is no submodule between them.

$$\iff \frac{M_{k+1}}{M_k} \text{ is simple}$$

$M$ has a finite $S'$-filtration for $S' = \text{The set of all simple modules}$
Example 2: Vector spaces

If \( R = k \) is a field and \( V \) is a vector space,

- Take a basis \( \beta \setminus \lambda \in k \) (infinite, \( k \) a cardinal)
- If \( V_\lambda = \langle v_\lambda : v \in \lambda \rangle \), then \( (V_\lambda : \lambda \in k) \)
  is a filtration of \( V \) with

\[
\dim_k \frac{V_{\lambda+1}}{V_\lambda} = 1.
\]

Every vector space is filtered by 1-dimenisonal vector spaces.
From $K(A)$ to $\text{mod-}R$

Neeman, Inv. Math, 2008

If we take a complex

$\cdots \rightarrow P_{n-1} \xrightarrow{d_{n-1}} P_n \xrightarrow{d_n} P_{n+1} \rightarrow \cdots$

$\Rightarrow \ker d^n$ is projective.

Remark 2.15. To illustrate the non-triviality of the implication (iii) $\Rightarrow$ (i) let us note a curious aside. Suppose $X$ is an acyclic chain complex

$\xrightarrow{\partial^{i-2}} X^{i-1} \xrightarrow{\partial^{i-1}} X^i \xrightarrow{\partial^i} X^{i+1} \xrightarrow{\partial^{i+1}}$

differential of projective modules. Suppose that, for each $i \in \mathbb{Z}$, the image $I^i$ of the differential $\partial^i : X^i \rightarrow X^{i+1}$ is a flat $R$-module.

By definition $X$ belongs to $K(R$-Proj), and by (iii) $\Rightarrow$ (i) $X$ also belongs to $K(R$-Proj)$^\perp$. Hence $X$ must be null homotopic. The module $I^i$, being a direct summand of $X^i$, is forced to be projective. What is curious about this aside is that the statement is the assertion that certain flat modules have to be projective; it does not mention triangulated categories. I do not know an elementary proof, a proof which avoids homotopy categories.
PERIODIC MODULES

Benson, Goodearl, Pac. J. Methy 2000

In an exact sequence in \text{Mod}-\mathcal{R},

\[ 0 \rightarrow N \xrightarrow{f} M \xrightarrow{g} N \rightarrow 0 \]

\( f \) monic
\( g \) epic
\( \text{Im } f = \text{ker } g \)

\( M \) projective, \( N \) flat \( \Rightarrow \) \( N \) projective

Remark

We can form the following exact co-ex

\[ \cdots \rightarrow M \xrightarrow{f} M \xrightarrow{g} M \xrightarrow{h} \cdots \]

\( s \nrightarrow N \xrightarrow{f} s \nrightarrow N \xrightarrow{f} \)

\( \text{ker } f = \text{Im } g \Rightarrow N \) flat
**PERIODIC MODULES**

**A-periodic modules**

Modules $M$ appearing in a short exact sequence

$$0 \rightarrow M \rightarrow A \rightarrow M \rightarrow 0$$

with $A \in \mathcal{A}$

**Main problem**

When $M$ belongs to $A$ as well?

Bazzoni, CI, Estrade, Alg. Rep, Th., 2020

If $\mathcal{Cot} = $ class of cotorsion modules then

Every $\mathcal{Cot}$-periodic module is cotorsion.
Totally acyclic complexes $X$

$$
\cdots \rightarrow X^{n-1} \xrightarrow{d^{n-1}} X^n \xrightarrow{d^n} X^{n+1} \rightarrow \cdots
$$

- $X^n$ is projective
- Acyclic: $\text{Ker} d^n = \text{Im} d^{n-1}$
- Totally acyclic

Gorenstein projective modules

If $M \cong \text{Ker} d^n$ for some totally acyclic complex $X$.

Proj $\subseteq$ GProj
Main open problem

Does there exists, for any module $M$, a morphism $f: G \rightarrow M$ with

- $f: G \rightarrow M$ with $f$ Gorenstein projective
- Lifting property

Good morphisms to make resolutions and "relative homological algebra"
Theorem (CI-Sasoch, 2022)

Set theory hypothesis

A cardinal $k$, $E, \lambda, \mu > k$ such that $\lambda$ is $\mu$-compact

$\Rightarrow$ Every module has a GP-precover for any ring.
**FROM K(A) TO Nod-R**


If \( X \) is coreflective in \( K(\text{Proj}) \), then every module has a Gorenstein-Projective precovers.

**Subcategory of** \( C(\text{Mod-R}) \)

\[ X = \text{All totally acyclic complexes.} \]

**Subcategory of** \( K(\text{Proj}) \)

\[ X = \text{The subcategory of } K(\text{Proj}) \text{ whose class of objects is } X. \]
Module category

- Facts on flat modules, countably generated, etc.

Homotopy category

- Neeman: $K(Proj)$ is $\mathcal{A}$$_1$-compactly generated

Schori-Stovicek

- $A$ is deconstructible $\implies$ $K(A)$ is coreflective

Every flat and Proj-periodic module is projective

Neeman

Some results in $K(Proj)$

Jørgensen

$\mathcal{X}$ is coreflective

Every module has a GP-precover.
OBJECTIVE OF THE TALK

Show recent results in this flavour

a) Study homotopy categories of $N$-complexes
   CI, Torrecillas, BMMS, 2023

b) Give new conditions that implies $X$ is
   coreflective in $\mathbf{K(Proj)}$
   CI, Sci. Chin. Math, 2023

\[\Downarrow\]

Every module has a $GP$-precover by
Jorgensen’s result.
2. Homotopy Categories of N-complexes

Fix $N \in \mathbb{N}$, $N \geq 2$

Category of N-complexes $X : C_N(R)$

- Objects: $\rightarrow X \xrightarrow{d_{n-1}} X \xrightarrow{d_n} X \xrightarrow{d_{n+1}} \cdots \xrightarrow{d_n \cdots d_{n-N+n}} = 0$

- Morphisms: Cochain maps.

Homotopy category of an additive subcategory $\mathcal{A} : K_N(A)$

- Objects: $N$-complexes in $\mathcal{A}$

- Morphisms: Homotopy equivalence classes
HOMOTOPY EQUIVALENCE

If $x, y \in C_n(R)$ and $f, g: X \to Y$ then

$f \sim g \iff \exists s^n: X \to Y$ such that

$$f^n - g^n = \sum_{j=0}^{N-1} d_Y^j s^{i+N-j-1} d_X^{N-j-1}$$

Case $N = 3$

$$f^n - g^n = d_{u-1} d_{u+1} u + d_u s d + s d u$$

Diagram:

- $X^{u-2}$ to $X^{u-1}$
- $X^u$ to $X^{u+1}$
- $X^{u+2}$
- $Y^{u-2}$ to $Y^{u-1}$
- $Y^u$ to $Y^{u+1}$
- $Y^{u+2}$
2.1. COREFLECTIVE SUBCATEGORIES OF $\mathbf{K}_n (\text{Mod-R})$

**Question:** How can we construct coreflective subcategories of $\mathbf{K}_n (\text{Mod-R})$?

*Following Saorín-Stovicek*

- We take a deconstructible class $\mathcal{A} = 1 - R$
- Construct subcategories of $\mathbf{K} (\text{Mod-R})$
  - $\mathbf{K}(\mathcal{A})$ (Remember: $N$-complexes from $\mathcal{A}$)
  - $E(\mathcal{A})$: $N$-acyclic complex from $\mathcal{A}$
$N$-ACYCLIC COMPLEXES

$X$ is $N$-acyclic if all $N$-homology modules vanish.

Case $N = 3$

Given an $N$-complex

$$
X \xrightarrow{d} X \xrightarrow{d} X \xrightarrow{d} X
$$

$d^u d^{u-1} d^{u-2} = 0 \Rightarrow \text{Im} d^{u-2} d^{u-1} \subseteq \text{Ker} d^u$

$d^{u+1} d^u d^{u-1} = 0 \Rightarrow \text{Im} d^{u-1} \subseteq \text{Ker} d^{u+1} d^u$

$X$ is $N$-acyclic if $\text{Im} d^{u-2} d^{u-1} = \text{Ker} d^u$ and $\text{Im} d^{u-1} = \text{Ker} d^{u+1} d^u$
2.1. Coreflective Subcategories

**Theorem**

If $A \subseteq \text{Mod-}R$ is decostructible $\Rightarrow K(A) \text{ and } E(A)$ are coreflective subcategories of $\mathcal{K}(\text{Mod-}R)$

**Proof**

1. $A$ decostructible $\Rightarrow$ $C(A)$ and $E(A)$ are decostructible in $\mathcal{C}(R)$
2. $\Rightarrow$ $C(A)$ and $E(A)$ are precovering in $\mathcal{C}(R)$
3. $\Rightarrow$ $K(A)$ and $E(A)$ are coreflective in $\mathcal{K}(\text{Mod-}R)$
2.1. Coreflective Subcategories

1. \( C(A) \) and \( E(A) \) are deconstructible.

If \( X \in C(A) \), we have to find some filtration of \( X \)

\[
\left( X_{\alpha} \mid \alpha < \kappa \right)
\]

\( X_0 : \ldots \to X_0^{n-1} \to X_0^n \to X_0^{n+1} \to \ldots \)

\( X_1 : \ldots \to X_1 \to X_1^n \to X_1^{n+1} \to \ldots \)

\( X_1 : \ldots \to X_1^{n-1} \to X_1^n \to X_1^{n+1} \to \ldots \)

Belonging to some set of complexes.

Deconstruction

Hill's lemma
2.1. Coreflective Subcategories

(2) $C^x(A)$ and $E^x(A)$ are precovering.

Theorem (Ershov, Saorín-Strickson)

Every deconstructible class is precovering.

Precovering class $\mathcal{X}$ in a category $C$

For every $C \in C$ exists a morphism $\eta : X \to C$ with

a) $X \in \mathcal{X}$

b) Lifting property

$\eta \circ f = \eta$

\[ X \xrightarrow{\eta} C \]

\[ f \]

\[ X' \in \mathcal{X} \]
2.1. Coreflective Subcategories

$K_n(A)$ and $E_n(A)$ are coreflective in $K_n(Mod-R)$

3.1 $K_n(A)$ and $E_n(A)$ are precovering in $K_n(Mod-R)$

Theorem (Cortés-Izurdiaga-Crivici-Saorín, 2022)

If $A \subseteq C$ is a subcategory of an additive category with split idempotents, then $A$ is coreflective if and only if:

a) $A$ is precovering.

b) $A$ is closed under direct summands.

c) Every morphism in $A$ has a pseudocokernel in $C$ which belongs to $A$. 
2.2. $\mathcal{K}_n(\text{Proj})$ is $\chi_1$-compactly generated.

**Theorem**

$\mathcal{K}_n(\text{Proj})$ is $\chi_1$-compactly generated.

**Proof**

The proof uses deconstruction and Hilbert lemma but with some particularities.

A proj is more than deconstructible: decomposable.

Kaplansky's Theorem

$P \in \text{Proj} \Rightarrow P = \bigoplus_{i=1}^{\infty} P_i$ with $P_i$ countably generated
2.2. $K_n(Proj)$ is $\aleph_1$-compactly generated

b) $C_{n}(Proj)$ is more than deconstructible

If $X \in C_{n}(Proj)$, then the filtration $(X_{a} | a \in k)$

\[
\begin{align*}
X_0 : & \quad \ldots \rightarrow X_0 \rightarrow X_0 \rightarrow X_0 \rightarrow \ldots \\
X_1 : & \quad \ldots \rightarrow X_1 \rightarrow X_1 \rightarrow X_1 \rightarrow \ldots \\
X_2 : & \quad \ldots \rightarrow X_2 \rightarrow X_2 \rightarrow X_2 \rightarrow \ldots \\
X_n : & \quad \ldots \rightarrow X_n \rightarrow X_n \rightarrow X_n \rightarrow \ldots \\
\end{align*}
\]

\[\xrightarrow{\text{split+}}\] morph.

\[\xrightarrow{\text{split+}}\] epim.

\[\text{countably generated projective}\]
**Splitting Morphisms**

**Split Monomorphism**

\[ M \xrightarrow{f} N \]

\[ sf(x) = x, \forall x \in M \]

\[ sf = 1_M \]

**Split Epimorphism**

\[ M \xrightarrow{f} N \]

\[ fs(x) = x, \forall x \in N \]

\[ fs = 1_N \]
3. Existence of GP-Precovers

- We work in $K(\text{Proj})$ and consider $K = \mathcal{X} \subseteq K(\text{Proj})$
- where $\mathcal{X}$ are the totally acyclic complexes.

Remember

\[ \cdots \rightarrow X^{n-1} \xrightarrow{d^{n-1}} X^n \xrightarrow{d^n} X^{n+1} \rightarrow \cdots \]

- $X^n$ is projective
- Acyclic: $\ker d^n = \text{Im} d^{n-1}$
- Totally acyclic

\[ X^{n-1} \xrightarrow{d^{n-1}} X^n \xrightarrow{d^n} X^{n+1} \]

$P \in \text{Proj}, f d^n = 0$
3. Existence of $GP$-Precovers

**Class of morphisms $\text{Morf}_K$**

- Take a morphism $f: X \to Y$ in $K(\text{Proj})$.
- By the axioms of triangulated categories

\[
X \xrightarrow{f} Y \to Z \to \Sigma X
\]

- Then $f \in \text{Morf}_K \iff Z \in K$.

**Relevance of $\text{Morf}_K$**

It allows to define the Verdier quotient

\[
K(\text{Proj})/K
\]
3. EXISTENCE OF GP-PRECOVERS

Theorem

If $\text{Mor}_K$ satisfies the generalized Baer lemma in $K(\text{Proj})$, then there exists GP-precovers in Mod-$R$.

Mor$_K$-injective objects

$X \in K(\text{Proj})$ is Mor$_K$-injective if

\[
\begin{array}{c}
\begin{array}{c}
A \rightarrow B \in \text{Mor}_K \\
\downarrow g \\
X \rightarrow X
\end{array}
\end{array}
\]

then $h \circ f = g$.
3. Existence of GP-Prelowers

Generalized Baer Lemma

A Mor Y satisfies the generalized Baer lemma if there is a set N ⊆ Mor Y (not a class!) such that

\[ X \text{ is } Mor Y \text{-injective} \iff X \text{ is } N \text{-injective} \]

Example

- \( M \in \text{Nod-}R \) is injective \( \iff \) any monomorphism

- Classical Baer lemma

  \( M \) is injective \( \iff \) \( M \) is \( N \)-injective

\[ N \parallel_1 \text{ Monomorphisms } f: I \to R \]
3. EXISTENCE OF GP-PRECOVERS

Theorem (CI-Guich-Kalebger-Srivastava, 2020)

If $(C; E)$ is an exact category satisfying certain conditions such that the class of injections satisfy the Generalized Baer Lemma, then $(C; E)$ has enough injectives.
3. **Existence of GP-Precovers**

**Theorem**

If $M$ is a module satisfying the generalized Baer lemma in $K(\text{Proj})$, then there exists GP-precovers in $\text{Mod-R}$.

**Proof**

a) The Verdier quotient $K(\text{Proj})/K$ has small hom-sets.

b) $K$ is coreflective in $K(\text{Proj})$.

Using results from triangulated categories.

C) There exist GP-precovers.

By Jorgensen's result.
3. Existence of GP-precovers

a) The Verdier quotient $K(\text{Proj})/K$ has small hom-sets.

- The category $K(\text{Proj})/K$
- Objects: $K(\text{Proj})$
- Morphisms between $X$ and $Y$
  Equivalent classes of triples $(Z,f,g)$

$$\exists \text{Hom}_{K}$$

It need not be a set!
REFERENCES


Thank you very much!