Generalized Gorenstein projective and flat modules

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Based on the following papers:
Definition

We say that a module $G \in Mod(R)$ is **Gorenstein projective** if there is an exact complex of projective modules

$$P = \ldots \rightarrow P_1 \xrightarrow{f_1} P_0 \xrightarrow{f_0} P_{-1} \rightarrow \ldots$$

such that $G = Z_0(P)$ and such that the complex stays exact when applying a functor $\text{Hom}(-, T)$, where $T$ is any projective module (i.e. the complex $\ldots \rightarrow \text{Hom}(P_{-1}, T) \rightarrow \text{Hom}(P_0, T) \rightarrow \text{Hom}(P_1, T) \rightarrow \ldots$ is exact for any projective module $T$).

Any projective module $P$ is Gorenstein projective ($0 \rightarrow P \xrightarrow{\text{Id}} P \rightarrow 0$)

Definition

We say that a module $M \in Mod(R)$ is **Gorenstein flat** if there is an exact complex of flat modules $F = \ldots \rightarrow F_1 \rightarrow F_0 \rightarrow F_{-1} \rightarrow \ldots$ such that $M = Z_0(F)$ and such that the complex stays exact when applying a functor $A \otimes -$, where $A$ is any injective module (i.e. the complex $\ldots \rightarrow A \otimes F_1 \rightarrow A \otimes F_0 \rightarrow A \otimes F_{-1} \rightarrow \ldots$ is exact for any injective module $A$).
A homomorphism $\phi : G \to M$ is a **Gorenstein projective precover** of $M$ if $G$ is Gorenstein projective and if for any Gorenstein projective module $G'$ and any $\phi' \in Hom(G', M)$ there exists $u \in Hom(G', G)$ such that $\phi' = \phi u$.

\[
\begin{array}{ccc}
  & G' & \\
  \downarrow{u} & \downarrow{h} & \\
  G & \rightarrow & M
\end{array}
\]

A precover $g : G \to M$ is said to be a **cover** if any homomorphism $u : G \to G$ such that $gu = g$, is an isomorphism.

A Gorenstein projective resolution of a module $M$ is a complex

\[
\ldots \rightarrow G_1 \xrightarrow{g_1} G_0 \xrightarrow{g_0} M \rightarrow 0
\]

such that $G_0 \to M$ and each $G_i \to Ker(G_{i-1} \to G_{i-2})$ for $i \geq 1$ are Gorenstein projective precovers.
Open question: the existence of the Gorenstein projective resolutions. Generalizations of the Gorenstein modules - the Ding projective modules

- The Ding projective modules are the cycles of the exact complexes of projective modules that remain exact when applying a functor $\text{Hom}(\cdot, F)$, with $F$ any flat module.

Open question: is the class of Ding projectives, $\mathcal{DP}$, precovering over any ring?
**FP<sub>n</sub>-injective and FP<sub>n</sub>-flat modules**

**Definition**

A module $M$ is \emph{n-finitely presented} (FP<sub>n</sub> for short) if there exists an exact sequence $F_n \to F_{n-1} \to \ldots \to F_1 \to F_0 \to M \to 0$ with each $F_i$ finitely generated free. A module $M$ is FP<sub>∞</sub> if and only if $M \in$ FP<sub>n</sub> for all $n \geq 0$.

$FP_0 \supseteq FP_1 \supseteq \ldots \supseteq FP_n \supseteq FP_{n+1} \supseteq \ldots \supseteq FP_\infty$, with FP<sub>0</sub> the class of all finitely generated modules, and FP<sub>1</sub> the finitely presented modules. A module $M$ is FP<sub>n</sub>-injective if $\text{Ext}_R^1(F, M) = 0$ for all $F \in$ FP<sub>n</sub>. From the definition, we get the following ascending chain:

$$\text{Inj} = \mathcal{IF}_0 \subseteq \mathcal{IF}_1 \subseteq \ldots \subseteq \mathcal{IF}_\infty.$$

A module $N$ is FP<sub>n</sub>-flat if $\text{Tor}_1(F, N) = 0$ for all $F \in$ FP<sub>n</sub>. From the definition, we get the following ascending chain:

$$\text{Flat} = \mathcal{IF}_0 = \mathcal{IF}_1 \subseteq \mathcal{IF}_2 \subseteq \ldots \subseteq \mathcal{IF}_\infty.$$
Definition

A module $G$ is Gorenstein $FP_n$-projective if it is a cycle in an exact complex of projective modules that remains exact when applying a functor $Hom(-, L)$ for any $L \in \mathcal{F}\mathcal{F}_n$. $\mathcal{GP}_n$ denotes the class of Gorenstein $FP_n$-projective modules.

We use $\mathcal{GP}_n$ to denote the class of Gorenstein $\mathcal{FP}_n$-projective modules.
- Since $\mathcal{F}\mathcal{F}_1 = Flat$, $\mathcal{GP}_1 = \mathcal{DP}$ (the Ding projective modules).
- And $\mathcal{F}\mathcal{F}_\infty = Level$, so $\mathcal{GP}_\infty = \mathcal{GP}_{ac}$ (the Gorenstein AC-projective modules).

By definition we have an ascending chain

$$\mathcal{GP}_\infty = \mathcal{GP}_{ac} \subseteq \cdots \subseteq \mathcal{GP}_2 \subseteq \mathcal{GP}_1 = \mathcal{DP} \subseteq \mathcal{GP}.$$ 

Main result for Gorenstein $FP_n$-projective modules:

**Theorem A**: Let $R$ be any ring. For any $n \geq 2$, $\mathcal{GP}_n$ is a precovering class.
A sufficient condition for a class \( C \) be precovering is to be the left half of a complete cotorsion pair.

Recall \( C^\perp = \{ M, \text{Ext}^1(C, M) = 0, \text{for all } C \in C \} \)
and \( \perp C = \{ L, \text{Ext}^1(L, C) = 0, \text{for all } C \in C \} \)

- A pair \(( C, \mathcal{L} )\) is a cotorsion pair if \( C^\perp = \mathcal{L} \) and \( \perp \mathcal{L} = C \).
- A cotorsion pair \(( C, \mathcal{L} )\) is complete if for every \( M \) there are short exact sequences \( 0 \to L \to C \to M \to 0 \) and \( 0 \to M \to L' \to C' \to 0 \) with \( C, C' \in C \) and with \( L, L' \in \mathcal{L} \).

A cotorsion pair \(( C, \mathcal{L} )\) is hereditary if \( \text{Ext}^i(C, L) = 0 \) for any \( C \in C \), any \( L \in \mathcal{L} \), all \( i \geq 1 \).

Examples: \((\text{Proj}, \text{Mod})\), \((\text{Mod}, \text{Inj})\).
Known: for $n \geq 2$, $M \in \mathcal{FF}_n \Leftrightarrow M^+ \in \mathcal{FI}_n$ (where $M^+ = \text{Hom}_\mathbb{Z}(M, \mathbb{Q}/\mathbb{Z})$) and $C \in \mathcal{FI}_n \Leftrightarrow C^+ \in \mathcal{FF}_n$.

So, for $n \geq 2$, $(\mathcal{FI}_n, \mathcal{FF}_n)$ is a duality pair in the sense of Bravo - Gillespie - Hovey.

**Theorem**

*(Bravo - Gillespie - Hovey)* Let $R$ be a ring and suppose $(\mathcal{C}, \mathcal{D})$ is a duality pair such that $\mathcal{D}$ is closed under pure quotients. Let $P$ be a complex of projective modules. Then $A \otimes P$ is exact for all $A \in \mathcal{C}$ if and only if $\text{Hom}(P, N)$ is exact for all $N \in \mathcal{D}$.

**Proposition**

A module $M$ is Gorenstein $FP_n$-projective if and only if there is an exact complex of projective modules $P = \ldots \rightarrow P_1 \xrightarrow{f_1} P_0 \xrightarrow{f_0} P_{-1} \rightarrow \ldots$ such that $M = Z_0(P)$ and such that $A \otimes P$ is exact for all $A \in \mathcal{FI}_n$. 
More general:

**Definition**

Let $\mathcal{B}$ be a fixed class of right $R$-modules. We say that a module $M$ is projectively coresolved Gorenstein $\mathcal{B}$-flat if $M = Z_0(P)$ for some $B \otimes -$-acyclic and exact complex $P$ of projective modules.

- $\mathcal{PGF}_\mathcal{B}$ denotes the class of projectively coresolved Gorenstein $\mathcal{B}$-flat modules.

**Question:** When is $\mathcal{PGF}_\mathcal{B}$ precovering?
- A class of modules $\mathcal{D}$ is *definable* if it is closed under direct products, direct limits and pure submodules. 

$(X$ is a pure submodule of $Y$ if there is a pure short exact sequence \[ \rho: 0 \rightarrow X \rightarrow Y \rightarrow Y/X \rightarrow 0 \]

i.e. an exact sequence such that the induced sequence \[ \text{Hom}_G(L, \rho): 0 \rightarrow \text{Hom}_G(L, X) \rightarrow \text{Hom}_G(L, Y) \rightarrow \text{Hom}_G(L, X/Y) \rightarrow 0 \]

in $\text{Ab}$ is exact for every finitely presented module $L$).

- The definable closure of $\mathcal{B}$, $< \mathcal{B} >$, is the smallest definable class containing $\mathcal{B}$.

- An *elementary cogenerator* of a definable class $\mathcal{D}$ is a pure-injective module $D_0 \in \mathcal{D}$ such that every $D \in \mathcal{D}$ is a pure submodule of some product of copies of $D_0$.

Here, *pure-injective* means injective with respect to pure exact sequences.

**Definition**

We say that a class $\mathcal{B}$ is **semi-definable** if it is closed under products and contains an elementary cogenerator of its definable closure.
Theorem

(joint with Estrada and Perez) If $\mathcal{B}$ is a semi-definable class of right $R$-modules then $(\mathcal{PGF}_\mathcal{B}, \mathcal{PGF}_\mathcal{B}^{\perp})$ is a complete hereditary cotorsion pair. In particular, the class $\mathcal{PGF}_\mathcal{B}$ is precovering.

Since for any $n > 1$ the class of $\mathcal{FP}_n$-injective modules, $\mathcal{FI}_n$, is definable (so semi-definable also), and since $\mathcal{GP}_n = \mathcal{PGF}_{\mathcal{FI}_n}$, we obtain:

Theorem

(Theorem A) Let $n \geq 2$. The class of generalized Gorenstein $\mathcal{FP}_n$-projective modules, $\mathcal{GP}_n$, is precovering.
Case $n = 1$

Lemma

$\mathcal{PGF} = \mathcal{DP} \cap \mathcal{GF}$.

Corollary

Over any ring $R$, $\mathcal{PGF} = \mathcal{DP}$ if and only if $\mathcal{DP} \subseteq \mathcal{GF}$.

Proposition

The Gorenstein flat dimension of a Ding projective module is either zero or infinite.

Proposition

The following are equivalent:

1. $\mathcal{DP} = \mathcal{PGF}$
2. Every Ding projective module has finite Gorenstein flat dimension.

Proposition

If $R$ has finite left weak Gorenstein global dimension then $\mathcal{DP} = \mathcal{PGF}$.
Theorem

Let $R$ be any ring. The following are equivalent:

1. $\mathcal{DP} \subseteq \mathcal{GF}$.
2. $\mathcal{DP} = \mathcal{PGF}$
3. For any Ding projective module $M$, its character module, $M^+$, is Gorenstein injective.
4. The class $\text{Inj}^+$ of all character modules of injective right $R$-modules, is contained in $\mathcal{DP}^\perp$.

Theorem

Let $R$ be a right coherent ring. Then $\mathcal{DP} = \mathcal{PGF} = \mathcal{GP}_{ac}$
The coherence is a sufficient condition, but it is not a necessary condition on the ring. If $R$ has finite global dimension (but it is not coherent) then $\mathcal{DP} = \mathcal{PGF}$.

Example. The ring

$$R = \begin{bmatrix} \mathbb{Q} & \mathbb{Q} & R \\ 0 & \mathbb{Q} & R \\ 0 & 0 & \mathbb{Q} \end{bmatrix} / \begin{bmatrix} 0 & 0 & R \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

is noncoherent of finite global dimension. So, $\mathcal{DP} = \mathcal{PGF}$ over $R$. 
Let $\mathcal{B}$ be a class of right $R$-modules. We say that a module $M \in Mod(R)$ is **Gorenstein $\mathcal{B}$-flat** if $M = Z_0(F)$ for some $(\mathcal{B} \otimes_R -)$-acyclic and exact complex $F$ of flat modules.

1. Gorenstein flat modules are obtained when $\mathcal{B} = \mathcal{I}nj$. If $\mathcal{B} \supseteq \mathcal{I}nj$ then any Gorenstein $\mathcal{B}$-flat module is, in particular, a Gorenstein flat module.

2. Recall that a module $M \in Mod(R)$ is **of type $FP_\infty$** if there exists an exact sequence

$$
\cdots \to P_1 \to P_0 \to M \to 0
$$

with $P_k$ finitely generated and projective for every $k \geq 0$. When $\mathcal{B} = \mathcal{F}\mathcal{I}_\infty = \mathcal{A}\mathcal{C} = (\mathcal{F}\mathcal{P}_\infty) \perp$ we obtain the class $\mathcal{G}\mathcal{F}_{\mathcal{A}\mathcal{C}}(R)$ of **Gorenstein $\mathcal{A}\mathcal{C}$-flat modules**.
Properties of Gorenstein $AC$-flat modules
1. $\mathcal{GF}_{AC}$ is a precovering class over any ring $R$.
2. If $\mathcal{GF}_{AC}$ is closed under extensions then $\mathcal{GF}_{AC}(R)$ is a covering class.

Remark. Our new results show that Gorenstein $AC$-flat modules are always closed under extensions, and so the latter two properties hold for any ring $R$. 
Properties of Gorenstein $\mathcal{B}$-flat modules

**Lemma**

Let $\mathcal{B}$ be a class of right $R$-modules. Then, the class $\mathcal{GF}_\mathcal{B}$ of Gorenstein $\mathcal{B}$-flat modules is a precovering class.

**Proposition**

If the class $\mathcal{GF}_\mathcal{B}$ of Gorenstein $\mathcal{B}$-flat modules is closed under extensions, then it is closed under taking kernels of epimorphisms and under direct limits. As a consequence, $\mathcal{GF}_\mathcal{B}$ is a covering class.

**Proposition**

If $\mathcal{GF}_\mathcal{B}$ is closed under extensions, then the pair $(\mathcal{GF}_\mathcal{B}, \mathcal{GC}_\mathcal{B})$ is a complete and hereditary cotorsion pair in $\text{Mod}(R)$, where $\mathcal{GC}_\mathcal{B}$ be the right orthogonal class $\mathcal{GF}_\mathcal{B}^\perp$.

Question: When is the class $\mathcal{GF}_\mathcal{B}$ closed under extensions?

We show that for any semi-definable class $\mathcal{B}$ we have $\mathcal{GF}_\mathcal{B} = \perp (\mathcal{C} \cap \mathcal{PGF}_\mathcal{B}^\perp)$, and so $\mathcal{GF}_\mathcal{B}$ is closed under extensions.
We use:

**Lemma**

The following are equivalent for any $R$-module $M$ and any class of right $R$-modules $\mathcal{B}$:

(a) $M$ is Gorenstein $\mathcal{B}$-flat.

(b) $\text{Tor}_i(B, M) = 0$ for all $i \geq 1$ and $B \in \mathcal{B}$, and there exists an exact and $(B \otimes \cdot)$-acyclic sequence of modules $0 \to M \to F^0 \to F^1 \to \ldots$ where each $F^i$ is flat.

(c) There exists a short exact sequence of modules $0 \to M \to F \to G \to 0$ where $F$ is flat and $G$ is Gorenstein $\mathcal{B}$-flat.
Theorem

Let $\mathcal{B}$ be a semi-definable class of right $R$-modules. Then, the following conditions are equivalent for every $M \in \text{Mod}(R)$:

(a) $M$ is Gorenstein $\mathcal{B}$-flat.

(b) There is a short exact sequence of modules

$$0 \to F \to L \to M \to 0$$

with $F \in \text{Flat}$ and $L \in \text{PGF}_\mathcal{B}$, which is also $\text{Hom}_R(-, C)$-acyclic, for any cotorsion module $C$.

(c) $\text{Ext}^1_R(M, C) = 0$ for every $C \in \mathcal{C} \cap \text{PGF}^\perp_\mathcal{B}$.

(d) There is a short exact sequence of modules

$$0 \to M \to F \to L \to 0$$

with $F \in \text{Flat}$ and $L \in \text{PGF}_\mathcal{B}$. 
(a) \( M \) is Gorenstein \( \mathcal{B} \)-flat.
(b) There is a short exact sequence of modules

\[ 0 \to F \to L \to M \to 0 \]

with \( F \in \mathcal{F}l_{\mathcal{B}} \) and \( L \in \mathcal{PGF}_{\mathcal{B}} \), which is also \( \text{Hom}_{\text{R}}(-, C) \)-acyclic, for any cotorsion module \( C \).

Proof of (a) \( \Rightarrow \) (b) \( M = Z_0(F) \), \( F \) an acyclic complex of flat modules, that is \( B \otimes - \) exact.

\((\text{dw}(\text{Proj}), (\text{dwProj})^\perp)\) is complete \( \Rightarrow \) exact \( 0 \to G \to P \to F \to 0 \), \( P \in \text{dw}(\text{Proj}) \), \( G \in (\text{dwProj})^\perp \).

Then \( G \) is flat.

\( F \) and \( G \) are \( \mathcal{B} \otimes - \) exact, so \( P \) is \( \mathcal{B} \otimes - \) exact.

Exact sequence \( 0 \to Z_iG \to Z_iP \to Z_iF \to 0 \) with \( Z_iG \) flat, and \( Z_iP \in \mathcal{PGF}_{\mathcal{B}} \).
If $C$ is a cotorsion module, both $g$ and $h$ are $C$-injective, so $f$ is also $C$-injective.
(b) There is a short exact sequence of modules

\[ 0 \to F \to L \to M \to 0 \]

with \( F \in \text{Flat} \) and \( L \in \text{PGF}_B \), which is also \( \text{Hom}_R(\cdot, C) \)-acyclic, for any cotorsion module \( C \).

(c) \( \text{Ext}^1_R(M, C) = 0 \) for every \( C \in \mathcal{C} \cap \text{PGF}^\perp_B \).

Proof of (b) \( \Rightarrow \) (c) Consider a short exact sequence as in (b).

\[ 0 \to F \to L \to M \to 0 \]

Let \( C \in \mathcal{C} \cap (\text{PGF}_B)^\perp \). We have an exact sequence

\[ \text{Hom}_R(L, C) \xrightarrow{\varphi} \text{Hom}_R(F, C) \to \text{Ext}^1_R(M, C) \to \text{Ext}^1_R(L, C) \]

where \( \text{Ext}^1_R(L, C) = 0 \) since \( L \in \text{PGF}_B \), and \( \varphi \) is epic. Hence, \( \text{Ext}^1_R(M, C) = 0 \).
(c) $\Ext_R^1(M, C) = 0$ for every $C \in \mathcal{C} \cap \mathcal{PGF}_B^\perp$.

(d) There is a short exact sequence of modules

$$0 \to M \to F \to L \to 0$$

with $F \in \mathcal{Flat}$ and $L \in \mathcal{PGF}_B$. 
(c) ⇒ (d): Consider a short exact sequence

\[ 0 \to M \to U \to T \to 0 \]

with \( U \in \mathcal{PGF}_B^\perp \) and \( T \in \mathcal{PGF}_B \). Let \( C \in \mathcal{PGF}_B^\perp \) be a cotorsion module. Then, we have an exact sequence

\[ Ext^1_R(T, C) \to Ext^1_R(U, C) \to Ext^1_R(M, C) \]

where \( Ext^1_R(T, C) = 0 \) and \( Ext^1_R(M, C) = 0 \). Then, \( U \in \perp (C \cap \mathcal{PGF}_B^\perp) \). Then \( U \) has a pure special \( \mathcal{PGF}_B \)-precovers.

- pure exact sequence

\[ 0 \to K \to L \to U \to 0 \]

with \( K \in \mathcal{PGF}_B^\perp \) and \( L \in \mathcal{PGF}_B \).

Then, \( L \in \mathcal{PGF}_B \cap (\mathcal{PGF}_B)^\perp \), so \( L \) is projective.

Then \( U \) is a pure epimorphic image of a projective module, so \( U \in \mathcal{F}lat \). (d) ⇒ (a): Follows from the Lemma above.
Corollary

If \( B \) is semi-definable then \( GF_B \) is closed under extensions.

Examples:
1. The class of Gorenstein flat modules is the left half of a complete hereditary cotorsion pair.
2. Consider the class \( GF_{AC} \) of Gorenstein AC-flat modules. The class \( AC \) of absolutely clean right \( R \)-modules is semi-definable. Hence, we have the following properties for Gorenstein AC-flat modules:
   - \( (GF_{AC}, (GF_{AC})^\perp) \) is a complete hereditary cotorsion pair.
   - Every module has a Gorenstein AC-flat cover.
Example 3. Consider the class $\mathcal{FI}_n$ of $FP_n$-injective right $R$-modules defined by Bravo-Perez. Recall that this class is the right orthogonal complement of that of the right $R$-modules of type $FP_n$, that is, those $N$ for which there is an exact sequence

$$P_n \rightarrow P_{n-1} \rightarrow \cdots \rightarrow P_1 \rightarrow P_0 \rightarrow N \rightarrow 0$$

where $P_k$ is finitely generated and projective for every $0 \leq k \leq n$. By Bravo-Perez, $\mathcal{FI}_n$ is a definable class if $n > 1$. Thus, if $\mathcal{GF}_{\mathcal{FI}_n}$ denotes the class of Gorenstein $\mathcal{FI}_n$-flat modules, we have that $\mathcal{GF}_{\mathcal{FI}_n}$ is closed under extensions. As a consequence of the previous results, we have that $(\mathcal{GF}_{\mathcal{FI}_n}, (\mathcal{GF}_{\mathcal{FI}_n})^{\perp})$ is a complete hereditary cotorsion pair.
All the results in this section are joint with S. Estrada and M. Perez

**The Gorenstein $\mathcal{B}$-flat stable model category**

Given two complete and hereditary cotorsion pairs $(\mathcal{Q}, \mathcal{R}')$ and $(\mathcal{Q}', \mathcal{R})$ in an abelian category $\mathcal{C}$ such that $\mathcal{Q}' \subseteq \mathcal{Q}$, $\mathcal{R}' \subseteq \mathcal{R}$ and $\mathcal{Q}' \cap \mathcal{R} = \mathcal{Q} \cap \mathcal{R}'$, then there exists a subcategory $\mathcal{W} \subseteq \mathcal{C}$ such that $(\mathcal{Q}, \mathcal{W}, \mathcal{R})$ is a Hovey triple in $\mathcal{C}$, that is:

1. $(\mathcal{Q}, \mathcal{R} \cap \mathcal{W})$ and $(\mathcal{Q} \cap \mathcal{W}, \mathcal{R})$ are complete cotorsion pairs in $\mathcal{C}$.
2. $\mathcal{W}$ is *thick*: it is closed under extensions, kernels of epimorphisms and cokernels of monomorphisms between its objects.

By Hovey’s correspondence, the existence of such a triple $(\mathcal{Q}, \mathcal{W}, \mathcal{R})$ implies the existence of a unique abelian model structure on $\mathcal{C}$ such that:

1. $\mathcal{Q}$ is the class of cofibrant objects.
2. $\mathcal{R}$ is the class of fibrant objects.
Let $\mathcal{B}$ be a class of modules that contains the injective right $R$-modules. We show it is possible to apply the previous result in the setting where:

$Q := \mathcal{GF}_B(R),$

$Q' := \mathcal{Flat}$ the class of flat left $R$-modules,

$\mathcal{R} := \mathcal{C} = (\mathcal{Flat})^\perp$ the class of cotorsion left $R$-modules,

$\mathcal{R}' := \mathcal{GC}_B,$

provided that $\mathcal{GF}_B(R)$ is closed under extensions (for instance if $\mathcal{B}$ is a semi-definable class).
Proposition (compatibility between the flat and Gorenstein $\mathcal{B}$-flat cotorsion pairs)

If $\mathcal{GF}_\mathcal{B}$ is closed under extensions and $\mathcal{B}$ contains all injective right $R$-modules, then

$$\mathcal{Flat} \cap \mathcal{C} = \mathcal{GF}_\mathcal{B} \cap \mathcal{GC}_\mathcal{B}$$

Proof. ($\supseteq$). Let $M \in \mathcal{GF}_\mathcal{B} \cap \mathcal{GC}_\mathcal{B}$. Then $M \in \mathcal{C}$. Since $M$ is Gorenstein $\mathcal{B}$-flat, we have a short exact sequence

$$0 \to M \to F \to M' \to 0$$

with $F$ flat, $M'$ is Gorenstein $\mathcal{B}$-flat. This sequence splits, since $M$ is Gorenstein $\mathcal{B}$-cotorsion, so $Ext^1(M', M) = 0$. Hence, $M$ is a direct summand of $F$, so $M \in \mathcal{Flat}$. 
Let \( N \in \mathcal{F} \cap \mathcal{C} \). Then \( N \in \mathcal{GF}_\mathcal{B} \). Since \((\mathcal{GF}_\mathcal{B}, \mathcal{GC}_\mathcal{B})\) is complete, there is a short exact sequence

\[
0 \to N \to C \to F \to 0
\]

with \( C \in \mathcal{GC}_\mathcal{B} \) and \( F \in \mathcal{GF}_\mathcal{B} \). Since \( N \) and \( F \) are Gorenstein \( \mathcal{B} \)-flat and \( \mathcal{GF}_\mathcal{B} \) is closed under extensions, we have that \( C \in \mathcal{GF}_\mathcal{B} \cap \mathcal{GC}_\mathcal{B} \subseteq \mathcal{F} \cap \mathcal{C} \). It follows that \( F \) is a Gorenstein flat module with finite flat dimension, and so \( F \) is flat. Then \( \text{Ext}^1(F, N) = 0 \) since \( N \) is cotorsion, and so the previous exact sequence splits. It follows that \( N \) is a direct summand of \( C \in \mathcal{GC}_\mathcal{B} \), and hence \( N \in \mathcal{GC}_\mathcal{B} \).
Thus we have:

**Theorem (the Gorenstein $\mathcal{B}$-flat model structure in $\text{Mod}(R)$)**

Assume $\mathcal{G}\mathcal{F}_\mathcal{B}$ is closed under extensions and $\mathcal{B}$ contains all injective right $R$-modules. Then, there exists a unique abelian model structure on $\text{Mod}(R)$ such that $\mathcal{G}\mathcal{F}_\mathcal{B}$ is the class of cofibrant objects.

**Corollary (the Gorenstein flat model structure over arbitrary rings)**

Over any ring $R$ there exists a unique abelian model structure on $\text{Mod}(R)$ such that $\mathcal{G}\mathcal{F}$ is the class of cofibrant objects.
Corollary

If $\mathcal{B}$ is a semi-definable class of right $R$-modules that contains the injectives, then, there exists a unique abelian model structure on $\text{Mod}(R)$ such that $\mathcal{GF}_\mathcal{B}$ is the class of cofibrant objects, $\mathcal{C}$ is the class of fibrant objects, and $\mathcal{PGF}_{\mathcal{B}}$ is the class of trivial objects.
References: