Jordan-Hölder exact categories

Representation Theory and Related Topics Seminar
at Northeastern University

Souheila Hassoun
J/W Thomas Brüstle and Aran Tattar
19 March 2021
PLAN

1. History of relative homology
2. Motivation
3. Exact categories
4. Jordan-Hölder exact categories
5. Examples and counter-examples
6. Admissible intersection and sum categories
7. General intersection, sum and radical
8. Artin-Wedderburn exact categories
9. The case of module categories over Nakayama algebras
10. Jordan-Hölder length function
THE ORIGINS OF RELATIVE HOMOLOGY
History of relative homology

1934 *Baer* introduced Ext for abelian groups

1940 *Baer* defines the Baer sum

1954 *Yoneda* proves *the classification theorem*, a one-to-one correspondence between the equivalence classes of the $n$–fold extensions of $B$ by $A$ and the elements of the abelian group $\text{Ext}^n(A, B)$

1955 *Buchsbaum* proves the existence of Ext for an exact category having enough projectives or enough injectives

1956 *Cartan and Eilenberg* generalize the notion extension groups

1957 *Buchsbaum* defines the extension functor $\text{Ext}$ without using the projective and the injective objects

1958 *Hochschild* discusses the analogous of the $\text{Ext}$ but applicable to a module theory that is *relativized* with respect to a given subring of the basic ring of operators

57-58 *Harrison* and *Heller* discuss similar problems, which make it natural to consider the extension functor on a specific exact categories
The idea of relative homological algebra for abstract categories is about the selection of a class of extensions or, equivalently, a class of monomorphisms and epimorphisms.
1961 *Butler and Horrocks* study relative homological algebras, but only for abelian categories.

They study how the derived functors behave under reduction of the exact structure.
Recent works:

1993 *Auslander and Ø.Solberg* discuss applying relative homological algebras to representation theory

1999 *Dräxler, Reiten, Smalø, Ø.Solberg + Keller* study the correspondence between exact structures and closed additive bifunctors of $\text{Ext}$

2005 *Auslander and Ø.Solberg* develop a general theory of relative cotilting modules for artin algebras
WHY
DO WE WANT TO STUDY THIS SUBJECT?
Motivation

Nice length function

Jordan-Hölder length improves [BHLR 18'] \(\mathcal{E}\)–length function
## Motivation

### Nice length function

Jordan-Hölder length improves \([\text{BHLR 18}']\) \(\mathcal{E}\)–length function

### Intersection and sum of subobjects

Jacobson radical, trace of subcategories, lattice of subobjects,...
## Motivation

**Nice length function**

Jordan-Hölder length improves [BHLR 18] $\mathcal{E}$–length function

**Intersection and sum of subobjects**

Jacobson radical, trace of subcategories, lattice of subobjects, ...

**New characterisations concerning additive categories**

It leads to new characterisations of the important and popular quasi-abelian (functional analysis) and abelian categories
Motivation

Nice length function

Jordan-Hölder length improves \([BHLR 18]\) \(E\)–length function

Intersection and sum of subobjects

Jacobson radical, trace of subcategories, lattice of subobjects,…

New characterisations concerning additive categories

It leads to new characterisations of the important and popular quasi-abelian (functional analysis) and abelian categories

Applications

The work by \([Berktas, Crivei, Kaynarca, Keskin, Tütüncü, 21]\) which generalize the theorems of uniqueness of uniform decompositions in abelian categories
## Motivation

<table>
<thead>
<tr>
<th>Nice length function</th>
</tr>
</thead>
<tbody>
<tr>
<td>Jordan-Hölder length improves [BHLR 18] $\mathcal{E}$–length function</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Intersection and sum of subobjects</th>
</tr>
</thead>
<tbody>
<tr>
<td>Jacobson radical, trace of subcategories, lattice of subobjects,...</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>New characterisations concerning additive categories</th>
</tr>
</thead>
<tbody>
<tr>
<td>It leads to new characterisations of the important and popular quasi-abelian (functional analysis) and abelian categories</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Applications</th>
</tr>
</thead>
<tbody>
<tr>
<td>The work by [Berktaş, Crivei, Kaynarca, Keskin, Tütüncü, 21] which generalize the theorems of uniqueness of uniform decompositions in abelian categories</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Jordan-Hölder property</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathcal{E}$ satisfies (JHP) if and only if the $\mathcal{E}$–Grenthendieck group is free</td>
</tr>
</tbody>
</table>
AXIOMATIC DEFINITION
Quillen’s exact categories

Definition

An **exact category** is a pair \((\mathcal{A}, \mathcal{E})\) consisting of an additive category \(\mathcal{A}\) and an exact structure \(\mathcal{E}\) on \(\mathcal{A}\).
Definition

An **exact structure** $\mathcal{E}$ on $\mathcal{A}$ is a class of kernel-cokernel pairs $(i, d)$ in $\mathcal{A}$ which is closed under isomorphisms and satisfies the following axioms:
Quillen’s exact categories

Definition

An **exact structure** $\mathcal{E}$ on $\mathcal{A}$ is a class of kernel-cokernel pairs $(i, d)$ in $\mathcal{A}$ which is closed under isomorphisms and satisfies the following axioms:

(A0) For all objects $A \in \text{Obj} \mathcal{A}$ the identity $1_A$ is an admissible monic and an admissible epic.
Quillen’s exact categories

**Definition**

An *exact structure* $\mathcal{E}$ on $\mathcal{A}$ is a class of kernel-cokernel pairs $(i, d)$ in $\mathcal{A}$ which is closed under isomorphisms and satisfies the following axioms:

(A0) For all objects $A \in \text{Obj}\mathcal{A}$ the identity $1_A$ is an admissible monic and an admissible epic.

(A1) The class of admissible monics (resp. admissible epics) is closed under composition.
(A2) The push-out of an admissible monic $i$ along an arbitrary morphism $a$ exists and yields an admissible monic $s_C$:

$$
\begin{array}{c}
A \xrightarrow{i} B \\
\downarrow a \quad \quad \quad \quad \quad \quad \quad \downarrow s_B \\
\downarrow a \\
C \xrightarrow{s_C} S \\
\end{array}
$$

(A2)' The pull-back of an admissible epic $h$ along $a$ exists and yields an admissible epic $p_B$:

$$
\begin{array}{c}
P \xrightarrow{p_B} B \\
\downarrow P_A \quad \quad \quad \quad \quad \quad \quad \downarrow a \\
A \xrightarrow{h} C. \\
\end{array}
$$
Remark

$(\mathcal{A}, \text{Ext}^1, \mathbb{I})$ is a Nakaoka-Palu *Extriangulated* category.
The lattice of exact structures

Theorems [BBGH, 7.34][BHLR, 5.4]:

Let \( \mathcal{A} \) be an additive category. The map \( \Phi : \mathcal{E} \mapsto \text{Ext}^1_{\mathcal{E}}(-, -) \) induces a lattice isomorphism between \((\text{Ex}(\mathcal{A}), \subseteq, \cap, \vee_{\text{Ex}})\) and \((\text{Cbf}(\mathcal{A}), \leq, \wedge, \vee_{\text{Cbf}})\).
Examples of exact structures

Smallest example

The minimal exact structure formed by all split short exact sequences.
Examples of exact structures

Smallest example
The minimal exact structure formed by all split short exact sequences.

[Largest example, Rump 2011]
There exists a unique maximal exact structure for any additive category $\mathcal{A}$. 
Additive categories

- additive
- weakly idempotent complete
- idempotent complete
- pre-abelian
- semi-abelian
- quasi-abelian
- abelian
Quasi-abelian definition

[RW77, Definition]:

A kernel \((A, f)\) is called *semi-stable* if for every push-out square

\[
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\downarrow t & & \downarrow s_B \\
C & \xrightarrow{s_C} & S \\
\end{array}
\]

the morphism \(s_C\) is also a kernel. We define dually a *semi-stable* cokernel. A short exact sequence \(A \xrightarrow{i} B \xrightarrow{d} C\) is said to be *stable* if \(i\) is a semi-stable kernel and \(d\) is a semi-stable cokernel. We denote by \(\text{sta}\) the class of all *stable* short exact sequences.

Definition

An additive category is *quasi-abelian* if it is *pre-abelian* and all kernels and cokernels are *semi-stable*.
Example

The maximal exact structure on a quasi-abelian category $\mathcal{A}$ consists of all short exact sequences on $\mathcal{A}$:

$$\mathcal{E}_{\text{max}} = \mathcal{E}_{\text{all}} = \mathcal{E}_{\text{sta}}.$$
JORDAN-HÖLDER PROPERTY
The Jordan-Hölder-Schreier theorem

**Theorem**

*If an $A$–module $X$ admits two composition series*

\[
0 = X_0 \subset X_1 \subset \cdots \subset X_{n-1} \subset X_n = X
\]

*and*

\[
0 = X'_0 \subset X'_1 \subset \cdots \subset X'_{m-1} \subset X'_m = X
\]

*then they are equivalent: $n = m$ and there exists a permutation $\sigma$ of $\{0, 1, \ldots, n - 1\}$ such that $X_{i+1}/X_i \cong X'_{\sigma(i)+1}/X'_{\sigma(i)}$.\]
## Notations

<table>
<thead>
<tr>
<th>Generalisation</th>
<th>Abelian categories</th>
<th>Exact categories</th>
</tr>
</thead>
<tbody>
<tr>
<td>subobjects $A \subseteq B$</td>
<td>$\mathcal{E}$–subobjects</td>
<td>$A \rightarrow B$</td>
</tr>
<tr>
<td>simple subobjects $0 \subset S$</td>
<td>$\mathcal{E}$–simple subobject</td>
<td>$0 \rightarrow S$</td>
</tr>
<tr>
<td>Composition series</td>
<td>$\mathcal{E}$–composition series</td>
<td></td>
</tr>
<tr>
<td>Jordan-Hölder property</td>
<td>$\mathcal{E}$–Jordan-Hölder property</td>
<td></td>
</tr>
<tr>
<td>Intersection, sum and Jacobson radical</td>
<td>New general intersection, sum and $\mathcal{E}$–radical</td>
<td></td>
</tr>
<tr>
<td>Artin-Wedderburn categories</td>
<td>$\mathcal{E}$–Artin-Wedderburn categories</td>
<td></td>
</tr>
<tr>
<td>Lattice of subobject</td>
<td>Poset of subobjects</td>
<td></td>
</tr>
</tbody>
</table>
$\mathcal{E}$—Jordan-Hölder property

**Definition**

Let $(\mathcal{A}, \mathcal{E})$ be an exact category. A finite $\mathcal{E}$—composition series for an object $X$ of $\mathcal{A}$ is a sequence

$$0 = X_0 \xrightarrow{i_0} X_1 \xrightarrow{i_1} \ldots \xrightarrow{i_{n-2}} X_{n-1} \xrightarrow{i_{n-1}} X_n = X$$

where all $i$ are *proper admissible monics* with $\mathcal{E}$—simple cokernel.
**Definition**

Let $(\mathcal{A}, \mathcal{E})$ be an exact category. A finite $\mathcal{E}$–composition series for an object $X$ of $\mathcal{A}$ is a sequence

\[
0 = X_0 \xrightarrow{i_0} X_1 \xrightarrow{i_1} \ldots \xrightarrow{i_{n-2}} X_{n-1} \xrightarrow{i_{n-1}} X_n = X
\]

where all $i$ are proper admissible monics with $\mathcal{E}$–simple cokernel. We say an $(\mathcal{A}, \mathcal{E})$ is a *Jordan-Hölder exact category* if any two finite $\mathcal{E}$-composition series of $X$ are equivalent.
Consider $(\mathcal{A}_S, \mathcal{E}_{min})$ with $\mathcal{A}_S = \{v \in \text{Vec}_k | \text{dim}_k(v) \in S = \mathbb{N} \setminus \{1\}\}$. 
Consider $(A_S, \mathcal{E}_{min})$ with $A_S = \{ v \in \text{Vec}_k | \text{dim}_k(v) \in S = \mathbb{N} \setminus \{1\} \}$.

There are two non-equivalent $\mathcal{E}_{min}$-composition series for the object $X = k^6$:

\[ 0 \rightarrow K^2 \rightarrow K^4 \rightarrow K^6 \]
Consider $A_S, E_{\text{min}}$ with $A_S = \{ v \in \text{Vec}_k | \text{dim}_k(v) \in S = \mathbb{N} \setminus \{1\} \}$.

There are two non-equivalent $E_{\text{min}}$-composition series for the object $X = k^6$:

$0 \rightarrow K^2 \rightarrow K^4 \rightarrow K^6$ with $\ell_{E_{\text{min}}}(k^6) = 2$.

$0 \rightarrow K^3 \rightarrow K^6$ with $\ell_{E_{\text{min}}}(k^6) = 1$.
Consider $\mathcal{R}e\mathcal{P}\mathcal{Q}$ with $\mathcal{Q}$:

$$1 \xrightarrow{\alpha} 2 \xleftarrow{\beta} 3$$
Consider $\text{Rep}\mathcal{Q}$ with $\mathcal{Q} : 1 \xrightarrow{\alpha} 2 \xleftarrow{\beta} 3$

The A-R sequences in $\text{Rep}\mathcal{Q}$ are

1. $S_2 \to P_1 \oplus P_3 \to l_2$
2. $P_3 \to l_2 \to S_1$
3. $P_1 \to l_2 \to S_3$. 
Examples and Counter-examples

The exact structures accordingly are
$E_{\text{min}}, E(1), E(2), E(3), E(1, 2), E(1, 3), E(2, 3), E_{\text{max}}$. 
The exact structures accordingly are
\( \mathcal{E}_{\text{min}}, \mathcal{E}(1), \mathcal{E}(2), \mathcal{E}(3), \mathcal{E}(1, 2), \mathcal{E}(1, 3), \mathcal{E}(2, 3), \mathcal{E}_{\text{max}}. \)

\((\text{Rep} Q, \mathcal{E}(1))\) is not Jordan-Hölder since there are non-equivalent \(\mathcal{E}(1)\)-composition series \(0 \to S_2 \to P_1 \oplus P_3\) and \(0 \to P_1 \to P_1 \oplus P_3.\)
Examples and Counter-examples

The exact structures accordingly are
\(E_{\min}, E(1), E(2), E(3), E(1, 2), E(1, 3), E(2, 3), E_{\max}\).

\((\text{Rep}_Q, E(1))\) is not Jordan-Hölder since there are non-equivalent
\(E(1)\)–composition series
\(0 \rightarrow S_2 \rightarrow P_1 \oplus P_3\) and
\(0 \rightarrow P_1 \rightarrow P_1 \oplus P_3\).

\((\text{Rep}_Q, E(2, 3))\) is not Jordan-Hölder since there are non-equivalent
\(E(2, 3)\)–composition series
\(0 \rightarrow P_1 \rightarrow I_2\) and
\(0 \rightarrow P_3 \rightarrow I_2\).
OUR APPROACH:
STUDY THE INTERSECTION AND SUM OF SUBOBJECTS
We call **A.I** the exact categories admitting **Admissible Intersections**: 

**[HR, Definition 4.3]**: An exact category \((\mathcal{A}, \mathcal{E})\) is called an **AI-category** if \(\mathcal{A}\) is pre-abelian satisfying: 

\((AI)\) The pull-back \((P, p_A, p_B)\) of two admissible monics \(a\) and \(b\) exists and yields two admissible monics:
[HR, Definition 4.4]:

An exact category $(\mathcal{A}, \mathcal{E})$ is called an **AS-category** if:

**AS**  The morphism $u$, given by the universal property of the push-out $(P, p_B, p_C)$ is an admissible monic.

![Diagram](image.png)
We call **A.I.S-categories** the exact categories admitting **Admissible Intersections and Sums**:

**[HR, Definition 4.5]:**

An exact category \((\mathcal{A}, \mathcal{E})\) is an **AIS-category** if it is an **Al-category** and moreover, the push-out along these pull-backs yields an admissible monic \(u\):

\[
\begin{align*}
P B & \rightarrow i \rightarrow B \\
\downarrow f & \quad & \downarrow l \\
C & \rightarrow k \rightarrow PO \\
\downarrow g & \quad & \downarrow j \quad \downarrow u \\
& \rightarrow D.
\end{align*}
\]
Quasi-abelian categories

[BHT, Theorem 4.12]:

Every AI-category $\mathcal{A}$ is quasi-abelian.
Quasi-abelian categories

[BHT, Theorem 4.12]:
Every AI-category \( \mathcal{A} \) is quasi-abelian.

[BHT, Theorem 4.17][HSW, Theorem 6.1 ]:
A category \( (\mathcal{A}, \mathcal{E}_{\text{max}}) \) is quasi-abelian if and only if it is an AI-category.
A pre-abelian category \( \mathcal{A} \) is quasi-abelian if and only if it has admissible intersections.

Proof

Let \( \mathcal{A} \) be a quasi-abelian additive category.

\[ \Rightarrow (\mathcal{A}, \text{ext}) \text{ is an exact category} \]

\[ \Rightarrow \text{the class of ad monics are precisely the class of kernels and every ad monic is the kernel of its cokernel} \]

We consider two ad monics (two ad subobjects) and we take their pull-back; it's the pull along kernels.
By [Kelly, 1969] $P_A$ and $P_B$ are also kernels.

So the P.B of ad monics is again ad and $(AI)$ is satisfied.

Let $(X, E)$ be an $(AI)$-exact category. Suppose that $E = E$ and all...
Proof

\[ S : o \rightarrow L \rightarrow M \rightarrow N \rightarrow o \]

We consider the following two sections (ad monics)

\[ S_0 (M, [\xi]) \cap E (M, [\eta]) = (L, \xi, \xi) \]

But \( \xi \) is not an ad monic!

\[ (\xi, \xi) \notin E \Rightarrow (A1) \]

by \([S6],[S9]\]

⇒ using (A2), (A3) or

\[ \Rightarrow \text{it is quasi-abelian by def.} \]

\[ E = \text{Eall} \]

\[ \text{Emin} \subseteq E \]
[BHT, Theorem 4.22]:

An exact category \((\mathcal{A}, \mathcal{E})\) is an AIS-category if and only if \(\mathcal{A}\) is abelian and \(\mathcal{E} = \mathcal{E}_{all}\).
Abelian categories

[BHT, Theorem 4.22]:

An exact category \((\mathcal{A}, \mathcal{E})\) is an AIS-category if and only if \(\mathcal{A}\) is abelian and \(\mathcal{E} = \mathcal{E}_{all}\).

[BHT, Theorem 3.7]

The following conditions are equivalent:

- \(\mathcal{A}\) is an abelian category,
- \((\mathcal{A}, \mathcal{E}_{all})\) is an AIS-category,
- \(\text{Hom}(\mathcal{A}) = \text{Hom}^{ad}(\mathcal{A})\),
- \(\text{Hom}^{ad}(\mathcal{A})\) is closed under composition,
- \(\text{Hom}^{ad}(\mathcal{A})\) is closed under addition.
We denote by $\mathcal{P}_{\mathcal{E}}^X$ the set of isomorphism classes of $\mathcal{E}$-subobjects of $X$. The relation $(Y, f) \leq (Z, g) \iff \exists Y \twoheadrightarrow f \twoheadrightarrow g \twoheadrightarrow Z \downarrow \downarrow \downarrow X$ turns $(\mathcal{P}_{\mathcal{E}}^X, \leq)$ into a poset.
Generalised Intersection

Definition

Consider two $\mathcal{E}$–subobjects $(A, f)$ and $(B, g)$ of $X$. We denote the set of all common admissible subobjects of $A$ and $B$ as

$$\text{Sub}_X(A, B) := \{ (Y, h) \in P^\mathcal{E}_X \mid Y \in P^\mathcal{E}_A, \ Y \in P^\mathcal{E}_B \}.$$
Consider two $\mathcal{E}$–subobjects $(A, f)$ and $(B, g)$ of $X$. We denote the set of all common admissible subobjects of $A$ and $B$ as

$$\text{Sub}_X(A, B) := \{ (Y, h) \in P^\mathcal{E}_X \mid Y \in P^\mathcal{E}_A, Y \in P^\mathcal{E}_B \}.$$ 

We define the $\mathcal{E}$–intersection of $(A, f)$ and $(B, g)$ in $P^\mathcal{E}_X$ as

$$l_X(A, B) := \text{Max}(\text{Sub}_X(A, B)).$$
Generalised Sum

Definition

we denote the set of all common superobjects of \( A \) and \( B \) as

\[
Sup_X(A, B) := \{ (Y, h) \in P^E_X \mid A \in P^E_Y, \ B \in P^E_Y \}
\]
**Definition**

we denote the set of all common superobjects of $A$ and $B$ as

$$\text{Sup}_X(A, B) := \{ (Y, h) \in P^\mathcal{E}_X \mid A \in P^\mathcal{E}_Y, B \in P^\mathcal{E}_Y \}$$

**[BHT, Definition 5.5]**

We define the $\mathcal{E}$—sum of $A$ and $B$ in $P^\mathcal{E}_X$ as

$$\text{Sum}_X(A, B) := \text{Min}(\text{Sup}_X(A, B)).$$
Examples

Let \((\mathcal{A} = \mathcal{V}_k^E, \mathcal{E}_{min})\) be the category of all even dimension \(k\)-vector spaces.
Examples

Let \((\mathcal{A} = \mathcal{V}_k^E, \mathcal{E}_{min})\) be the category of all even dimension \(k\)-vector spaces.

Consider the object \(X = k^6 = \langle v_1, v_2, v_3, v_4, v_5, v_6 \rangle\) and

\[ V_1 = \langle v_1, v_2, v_3, v_4 \rangle \quad \text{and} \quad V_2 = \langle v_2, v_3, v_4, v_5 \rangle. \]
Examples

Let \( \mathcal{A} = \mathcal{V}_k^E, \mathcal{E}_{\text{min}} \) be the category of all even dimension \( k \)-vector spaces.

Consider the object \( X = k^6 = \langle v_1, v_2, v_3, v_4, v_5, v_6 \rangle \) and

\[
V_1 = \langle v_1, v_2, v_3, v_4 \rangle \quad \text{and} \quad V_2 = \langle v_2, v_3, v_4, v_5 \rangle .
\]

The abelian intersection \( V_1 \cap V_2 = \langle v_2, v_3, v_4 \rangle \) in \( \text{mod } k \)

**BUT**

\[
\text{Int}_X^{\mathcal{E}_{\text{min}}}(V_1, V_2) = \text{Gr}(2, 3)
\]

and

\[
\text{Sum}_X^{\mathcal{E}_{\text{min}}}(V_1, V_2) = \{(X, \mathbb{I})\}.
\]
We define the \( \mathcal{E} - \)Jacobson radical to be the generalised intersection

\[
\text{rad}_\mathcal{E}(X) := \bigcap_{(Y, f) \in S_X} \{ (Y, f) \in \text{Max}(S_X) \}.
\]
We define the $\mathcal{E}$—Jacobson radical to be the generalised intersection

$$\text{rad}_\mathcal{E}(X) := l_X \{ (Y, f) \in S_X \mid (Y, f) \in \text{Max}(S_X) \}.$$ 

**[BHT, Proposition 6.2]**

For all $X, Y \in$ and $R \xrightarrow{r} X$.
- For all $(R, r) \in \text{rad}(X)$, $\text{rad}(\text{Coker}(r)) = \{0\}$. 
We define the $\mathcal{E}$–Jacobson radical to be the generalised intersection
\[
\text{rad}_\mathcal{E}(X) := I_X \{ (Y, f) \in S_X \mid (Y, f) \in \text{Max}(S_X) \}.
\]

For all $X, Y \in$ and $R \xrightarrow{r} X$.
- For all $(R, r) \in \text{rad}(X)$, $\text{rad} (\text{Coker}(r)) = \{0\}$.
- For all $(Z, g) \in S_X$, $Z$ is an $\mathcal{E}$-subobject of some $(R, r) \in \text{rad}(X)$ if and only if $pg = 0$ for all $\mathcal{E}$–simple quotients $p : X \twoheadrightarrow S$ of $X$. 
An exact structure $\mathcal{E}$ on $\mathcal{A}$ is called Artin-Wedderburn if for any object $X \in \mathcal{A}$ the following properties are equivalent:

(AW1) Every sequence in $\mathcal{E}$ of the form $A \rightarrow X \rightarrow X/A$ splits,

(AW2) $X$ is $\mathcal{E}$—semisimple,

(AW3) $\text{rad}(X) = \{0\}$.
Artin-Wedderburn exact structures

[BHT, Definition 6.4]:

An exact structure $\mathcal{E}$ on $\mathcal{A}$ is called Artin-Wedderburn if for any object $X \in \mathcal{A}$ the following properties are equivalent:

(WA1) Every sequence in $\mathcal{E}$ of the form $A \rightarrow X \rightarrow X/A$ splits,

(WA2) $X$ is $\mathcal{E}$—semisimple,

(WA3) $\text{rad}(X) = \{0\}$.

We say in this case that $(\mathcal{A}, \mathcal{E})$ is an $\mathcal{E}$—Artin-Wedderburn category.
The split exact structure $\mathcal{E}_{\text{min}}$ is an Artin-Wedderburn exact structure:

[BHT, lemma 6.7]:

Any additive category $\mathcal{A}$ is an $\mathcal{E}_{\text{min}}$-Artin-Wedderburn category.
A category is **Idempotent complete** if every idempotent splits.

**Definition**

A **Krull-Schmidt** category is a Hom-finite and Idempotent complete additive category.
**Definition**

A category is *idempotent complete* if every idempotent splits.

**Definition**

A *Krull-Schmidt* category is a Hom-finite and Idempotent complete additive category.

**[BHT, Theorem 6.8]:**

Let \((\mathcal{A}, \mathcal{E})\) be a Krull-Schmidt \(\mathcal{E}\)-Artin-Wedderburn category. Then \((\mathcal{A}, \mathcal{E})\) is a Jordan-Hölder exact category.
Definition

An uniserial module $M$ is a module over a ring, such that $\mathcal{P}_M^{\mathcal{E}_{\text{max}}}$ is a totally ordered set.

Definition

A finite-dimensional algebra $\Lambda$ is called Nakayama if every indecomposable right and left projective $\Lambda$-module is uniserial.
Nakayama Algebras

Definition

An uniserial module $M$ is a module over a ring, such that $\mathcal{P}_M^{\mathcal{E}_{\text{max}}}$ is a totally ordered set.

Definition

A finite-dimensional algebra $\Lambda$ is called Nakayama if every indecomposable right and left projective $\Lambda$-module is uniserial.

[BHT, Theorem 6.13]:

Let $\Lambda$ be a Nakayama algebra, and denote $\mathcal{A} = \text{mod}\Lambda$. Then an exact category $(\mathcal{A}, \mathcal{E})$ is $\mathcal{E}$–Artin-Wedderburn precisely when it is Jordan-Hölder.
We define the $\mathcal{E}$–Jordan-Hölder length $l_{\mathcal{E}_{JH}}(X)$ of an object $X$ as the length of an $\mathcal{E}$–composition series of $X$. That is $l_{\mathcal{E}_{JH}}(X) = n$ if and only if there exists an $\mathcal{E}$–composition series

$$0 = X_0 \rightarrow X_1 \rightarrow \ldots \rightarrow X_n = X$$

We say in this case that $X$ is $\mathcal{E}$–finite.
We define the $E$–Jordan-Hölder length $l_{EJH}(X)$ of an object $X$ as the length of an $E$–composition series of $X$. That is $l_{EJH}(X) = n$ if and only if there exists an $E$–composition series

$$0 = X_0 \rightarrow X_1 \rightarrow \ldots \rightarrow X_n = X$$

We say in this case that $X$ is $E$–finite.

Clearly, isomorphic objects have the same length, and therefore this definition gives rise to a length function $l_E : ObjA \rightarrow \mathbb{N} \cup \{\infty\}$ defined on isomorphism classes.
[BHT, Corollary 7.2]:

Let

\[ X \hookrightarrow Z \twoheadrightarrow Y \]

be an admissible short exact sequence of finite length objects. Then

\[ l_{\mathcal{E}}(Z) = l_{\mathcal{E}}(X) + l_{\mathcal{E}}(Y). \]
Properties of $l_{\mathcal{E}_{JH}}$

[BHT, Corollary 7.2]:

Let

$$X \hookrightarrow Z \twoheadrightarrow Y$$

be an admissible short exact sequence of finite length objects. Then

$$l_{\mathcal{E}}(Z) = l_{\mathcal{E}}(X) + l_{\mathcal{E}}(Y).$$

[BHT, Proposition 7.8]:

If $\mathcal{E}$ and $\mathcal{E}'$ are exact structures on $\mathcal{A}$ such that $\mathcal{E}' \subseteq \mathcal{E}$, then $l_{\mathcal{E}'}(X) \leq l_{\mathcal{E}}(X)$ for all objects $X$ in $\mathcal{A}$. 
An object $X$ of $(\mathcal{A}, \mathcal{E})$ is $\mathcal{E}$–Artinian and $\mathcal{E}$–Noetherian if and only if it has an $\mathcal{E}$–finite length.
THANK YOU!