Descending chains of coprime pairs
and the exchange property

P. A. Guil Asensio, Univ. of Murcia
(joint work with M. C. Izurdiaga)
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I. a. Right cotorsion rings

**Def.** - A module $M_R$ is cotorsion if $\text{Ext}^1_R(F, M) = 0$ for all flat $F_R$.

Cotorsion modules were introduced by Harrison in 1959 as a homological generalization of pure-injective rings.

For instance, in abelian groups, they are just the (not necessarily pure) quotients of pure-injective groups.
**Def.** A ring $R$ is right cotorsion if so is $RR$

**Facts**

1. $R_R$ is cotorsion iff it is the endomorphism ring of a flat cotorsion module.
2. Flat cotorsion modules in $\text{Mod}-R$ are just the pure-injective objects in $\text{Flat}-R$ (but not necessarily in $\text{Mod}-R$)
Def. Let $A$ be an additive category with direct limits. An object $A \in A$ is called finitely presented if $\text{Hom}(A,-)$ commutes with direct limits.

Def. An additive category $A$ with direct limits is called finitely accessible if there exists a set $A_0$ of finitely presented objects of $A$ such that $A = \varinjlim A_0$. 
(Finitely) accessible categories have been studied, for instance, by Adámek, Crawley-Boevey, Makkai, Paré, Prest, Rosicky, ...

Prop. Any additive finitely accessible category is equivalent to \( \text{Flat} \cdot R \) for some ring with enough idempotents \( R \).

Moreover, the solution to the flat cover conjecture gives as a byproduct:
Theorem (El Bashir, Enochs, Bican)

For every right module $M$, there exists a monomorphism $u: M \rightarrow C$ into a cotorsion module $C$ such that:

i) Any other morphism $f: M \rightarrow C'$, with $C'$ cotorsion, factors through $u$.

ii) $u$ is minimal, in the sense that if $f: C \rightarrow C$ satisfies that $fou = u$, then $f$ is an automorphism.
Corollary. Define that
0 → A → B → C → 0
is pure-exact in an additive finitely accessible A
category when it is a direct limit of splitting sequence.
Then we get:

Corollary. Any object has a pure-injective envelope
in A. Moreover, its endomorphism ring is
a right cotorsion ring.
Problem. Study the structure of these rings.

Difficulty. Usual techniques in purity, such as Functor Categories, finite elimination subgroups, or Model Theory seem to fail in this setup.

Example. Let $R$ be a right perfect ring which is not right pure-injective.
$R_R$ is trivially (Σ-)cotorsion, since every flat right module is projective.

Assume it would be possible to construct a full embedding

$$U : \text{Flat}-R \longrightarrow \mathcal{C}$$

in a Grothendieck category $\mathcal{C}$ so that $R$ becomes injective in $\mathcal{C}$. Then:

$$R'_R = \text{End}_R(R) = \text{End}_\mathcal{B}(U(R))$$
and $R_0$ would be right p.i., since endomorphism rings of injective objects in Grothendieck categories are pure-injective. But there exist non right p.i. right cotorsion rings.

Idea (G. Herzog) Endomorphism rings of pure-injective modules are (von Neumann) regular and right self-inj. modules their Jacobson radical. And regular rings are plenty of idempotents.

Develop a method to construct idempotents in right cotorsion rings.
Theorem (G. Gómez Paredes). Let $M$ be an indecomposable module and assume any directed union of direct summands of $M(I)$ is a direct summand for any set $I$. Then $\text{End}(M)$ is local.

Application:

- Decomposition of modules into direct summands
- Recently used by Positselski and Stovicek to study contra-modules associated to cosheaves.
In our setting: choose an element $a \in R$.

- Clearly $Ra + R(1-a) = R$. So if $Ra \cap R(1-a) = 0$, then $R = Ra \oplus R(1-a)$ and we got our idempotent.

- Otherwise, $Ra^2 \subseteq Ra$ and $R(1-a^2) \subseteq R(1-a)$ and again $R = Ra^2 + R(1-a^2)$. We may hope that $Ra^2 \cap R(1-a^2) \subseteq Ra \cap R(1-a)$. 
Continuing in this fashion, we get commutative diagrams with splitting rows,

\[
\begin{array}{ccc}
\mathbb{R}^n & \xrightarrow{(a, 1-a)} & \mathbb{R}^n \\
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\| & & \| \\
\mathbb{R}^n & \xrightarrow{(a, 1-a)} & \mathbb{R}^n \\
\end{array}
\]

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\begin{array}{ccc}
\mathbb{R}^n & \xrightarrow{(a^i, 1-a^i)} & \mathbb{R}^n \\
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whose direct limit would be of the form

\[ \Sigma = R \xrightarrow{u} \bigoplus F_i \xrightarrow{G} \bigoplus F_j \]

with $F_i$, $F_j$, $G$ flat. So it splits since $R$ is coherent.

Call $\pi : \bigoplus F_i \rightarrow R$ the splitting. Then, if $(x, y) = u(1)$, call $aw = \pi(x)$, $bw = \pi(y)$.

Then $Raw + Rbw = R$. And thus, we have a splitting sequence

\[ R \xrightarrow{(aw, bw)} R \oplus R \]

below the diagram.
Continue transfinitely in this fashion and show that the process stops by cardinality. And show that when the process stops, say in \((a_0, b_0)\), the \(R = R_{a_0} \circ R_{b_0}\).

We may formalize this idea as follows:

**Def.** A pair of elements \(a, b \in R\) are called a least coprime pair if \(R_a + R_b = R\). Denote it by \(<a, b>\).

Our prototype is \(<a, 1-a>\).
We will say that \( \langle a, b \rangle \leq \langle a', b' \rangle \) if \( Ra \subseteq Ra' \) and \( Rb \subseteq Rb' \).

**Theorem** (C., Henze). Let \( R \) be a right cotorsion ring. And let \( \langle a, b \rangle \) be a left coprime pair. Then there exists an idempotent \( e \in R \) s.t. \( \langle e, 1-e \rangle \leq \langle a, b \rangle \).

Moreover, \( \langle e, 1-e \rangle \) is minimal in the former order relation.
Following these ideas, we were able to prove:

**Theorem** Let $R$ be a right cotorsion ring. Then $R/J$ is von Neumann regular and right self-injective and idempotents lift modulo $J$.

Indeed, our results suggest a possible extension of Ziegler spectrum to additive finitely accessible categories.
**Definition**

Let \( \{X_i\}_I \) be a family of modules and \( u: M \to \prod \limits_I X_i \) an embedding. \( u \) is strongly pure if for any \( R \in I \), the canonical morphism

\[
M \otimes_R L \to (\prod \limits_I X_i) \otimes L \to \prod \limits_I (X_i \otimes L)
\]

is a monomorphism.

**Theorem** (G. Rothmaler) A submodule \( M \) of a product of flat modules is flat iff it is strongly pure.
Theorem (C. Herzog). There exist a set \( \mathcal{I} \) of indecomposable pure-injective objects in \( \text{Flat}_R \) s.t. a right \( R \)-module \( M \) lives in \( \text{Flat}_R \) iff it is a strongly pure submodule of a product of them.

Remark The proof of the theorem is inspired by Awlander's objects defined by functors.
II.6. Exchange rings

Def. (Crawley and Jonsson ’64, Warfield Jr. ’72)

A right $R$ module $M$ satisfies the (finite) exchange property if for any module $X$ and decompositions

$$X = M' \oplus Y = \bigoplus_{i \in I} N_i,$$

with $M' \cong M$ (resp., $I$ finite), there exist submodules $N'_i \leq N_i$ s.t.

$$X = M' \oplus \left( \bigoplus_{i \in I} N'_i \right).$$
II. Exchange rings

Theorem. (Nicholson ’77) $R_R$ satisfies the (finite) exchange property iff so does $R_R$.

Theorem (Nicholson ’77, Goodearl ’76) $R_R$ satisfies the finite exchange property iff $\forall e \in R$ $Ye = e^2 e R$

s.t. $e R \subseteq a R$ and $(1 - e) R \subseteq (1 - a) R$

Def.- These rings are called exchange rings.
Thus, our construction was implicitly giving a method to prove that a ring is exchange.

However, there are many exchange rings which are not von Neumann. So it would be interesting to study:

I) For which exchange rings, the above construction can be done.

II) What additional properties we get.
Example of exchange rings

I) Right self-injective rings
II) Right pure-injective rings
III) Right continuous rings
IV) Local rings
V) Right perfect rings.
   In particular, right artinian rings.

VI) (von Neumann) regular rings
VII) Sempiperfect rings
VIII) Semiregular rings
IX) Right cotorsion rings
X) Rings invariant under automorphisms of their injective envelope.
**Theorem.** (Warfield) $M_R$ satisfies the finite exchange property iff $\text{End}_R(M)$ is an exchange ring.

**Main open question**

Does the finite exchange property in $M_R$ imply the general one? True if $M_R$ is finitely generated.

Open for almost 60 years.
III. Descending chains of coprime pairs

Def. - Let $\mathbb{F}=1<\alpha_1, \beta_1>$ be a descending chain of right coprime pairs. We will say that $\mathbb{F}$ is compatible if there exist pairs of scalars $(r_{\alpha}, r_{\beta}, s_{\alpha})$ for every $\alpha \leq \beta < \gamma$ s.t.

i) $a_{\beta} = r_{\alpha_{\beta}} a_{\alpha}$ and $b_{\beta} = s_{\alpha_{\beta}} b_{\alpha}$, if $\alpha \leq \beta$

ii) $r_{\alpha_{\delta}} = r_{\beta_{\delta}} r_{\alpha_{\beta}}$ and $s_{\alpha_{\delta}} = s_{\beta_{\delta}} s_{\alpha_{\beta}}$, if $\alpha \leq \beta \leq \delta < \gamma$
Def. A ring $R$ is right strongly exchange if for any compatible descending chain of right coprime pairs $S$, there exist a minimal right coprime pair seated below it.

Example:
1. Any left self-injective ring is right strongly exchange.
II. Any left pure-injective ring is right strongly exchange.

In particular \( \text{End}_R(E) \) is right strongly exchange for any (pure-) injective or quasi-injective module \( M \).

III. Any local ring is right strongly exchange.
III. Let $V$ a vector space with $\dim V = n$. Then $S = \text{End}_k(V)$ is regular and right self-injective. Therefore, it is left strongly exchange. However, it is not right strongly exchange.

Let $B = \{v_1, \ldots, v_n, \ldots \}_{n \in \mathbb{N}}$ be a basis and call $\alpha_i : V \to V$ the endomorphisms $\alpha_i(v_j) = \delta_{ij} v_j \quad \forall j \in \mathbb{N}$.
Let \( p : \mathbb{N} \to \mathbb{N} \) defined as \( p(n) = \bigcup_{i \in \mathbb{N}} \forall i \in \mathbb{N} \).
Then \( \bigcap_{i=1}^{n} (1 - \sum_{i=1}^{n} e_i, 1 - p) \neq 0 \) is a descending chain of right coprime pairs which do not have any minimal one below it.

In particular:

- Strongly exchange is not a left-right symmetric condition
- Regular rings don't need to satisfy it.
IV. Any right torsion ring is left strongly exchange.

V. A module $N_R$ is said continuous if:
   i) Any $N \leq N_R$ essentially embeds in a direct summand of $M$.
   ii) If $N \leq M_R$, it is isomorphic to a direct summand of $M_R$, then $N \leq_\oplus M$. 
The notion of continuous module has its origin in von Neumann continuous geometries and play an important role in the structure of (von Neumann) regular rings.

Indeed, in regular rings, condition II is always satisfied

- Any right continuous ring is left strongly exchange.
VI. Left perfect rings are right strongly exchange.

Note.- All the above classes of rings satisfy that $R/J$ is regular and idempotents lift modulo $J$.

This suggests that this might be a general property of strongly exchange rings.
Note. All the above classes of rings satisfy that $R/J$ is regular and idempotent lift modulo J.

This suggests that this might be a general property of strongly exchange rings. This would allow to get a unified study of them.
IV. Structure of right strongly exchange rings

The following result is clear.

**Prop.** Any right strongly exchange ring is an exchange ring.

**Theorem** (G. Izurdiaga) Let \( R \) be a right strongly exchange ring. Then \( R/J \) is regular and idempotents lift modulo \( J \).
Idea of the proof

• Characterize first $J$ as the elements $a \in R$ s.t. the only minimal right coprime pair below $<a, 1-a>$ is $<0, 1>$.

• Now use the existence of minimal coprime pairs to show that $R/J$ is regular by constructing idempotents. This is the hardest part.

• Finally, show that idempotents lift. □
Remark: If $R_p$ is continuous, then $R/I$ does not need to be right self-injective.

Therefore:

- Right strongly exchange rings do not need to be left self-injective modulo their Jacobson radical.
- However, if $R$ is left continuous, so is $R/I$.
• We do not know whether a right strongly exchange ring \( R \) satisfies that \( R/I \) is left continuous.

• However, this is the case if we strengthen a little bit the definition. But then, we lose some example of right strongly exchange rings.
**Theorem.** Assume that for every descending system of compatible right coprime pairs, there exists a minimal right coprime pair below it. Then $R/I$ is left continuous.

**Note.** This is also the case for countable right strongly exchange rings. (even if we only assume that they have countable uniform dimension)
VI Open Questions

Let $R$ be a right strongly exchange ring. Does it satisfy the above condition?

I.e., does there exist a minimal right coprime pair below any compatible descending system of right coprime pairs?
II. Let $R$ be a right strongly exchange ring. Is $R/J$ left continuous?

III. Recently, Lam, Khureva and Nielsen have proved that if $R$ is an exchange ring, then for any element $a \in R$, the idempotent below $\langle a, 1-a \rangle$ can be chosen with a "two-sided property", in the sense that:

$e \in aR \cap Ra$ and $1-e \in (1-a)R \cap R(1-a)$
The fact that being strongly exchange is not a left-right symmetric property shows that we cannot expect a similar result.

However, proofs frequently involve some left-right symmetric properties. It would be interesting to clarify this point.
IV. Exchange rings are related to the study of the unimodular equation
\[ ax + by = 1 \]
For instance, for square matrices.
It would be interesting to study infinite systems of unimodular equations in terms of strongly exchange properties.