Interpretation Functors II

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Let $R$ be a ring. A **pp-$n$-formula** (over $R$) is a formula

$$\varphi(\overline{x}) := \exists \overline{y} \ (\overline{x} \ \overline{y})A = 0$$

where $A$ is an $(n + m) \times l$ matrix with entries in $R$, $\overline{x}$ an $n$-tuple of variables and $\overline{y}$ an $m$-tuple of variables. Write $\text{pp}_R^n$ for the set of pp-$n$-formulae over $R$.

For $M$ an $R$-module and $\varphi \in \text{pp}_R^n$, we write $\varphi(M)$ for the solution set of $\varphi$ in $M$.

A full subcategory $\mathcal{D} \subseteq \text{Mod-}R$ is called a **definable subcategory** if it is the form

$$\mathcal{D} = \{ M \in \text{Mod-}R \mid \varphi_i(M) = \psi_i(M) \text{ for all } i \in I \}$$

where $\varphi_i, \psi_i$ are pairs of pp-formulae indexed by $I$. 
Interpretation functors

Definition
Let $R$, $S$ be rings and $\mathcal{D}$ a definable subcategory of $\text{Mod}-S$. Suppose that $\varphi, \psi$ are pp-$n$-formulae over $S$ and that for each $r \in R$, $\rho_r(\overline{x_1}, \overline{x_2})$ is a pp-$2n$-formula in variables $\overline{x_1}, \overline{x_2}$ each of length $n$.

Suppose that for all $M \in \mathcal{D}$ the following hold:

1. $\varphi(M) \supseteq \psi(M)$
2. for all $r \in R$, $\rho_r(\overline{x_1}, \overline{x_2})$ defines an endomorphism $\rho^M_r$ of the abelian group $\varphi(M)/\psi(M)$
3. $\varphi(M)/\psi(M)$ equipped with the $\rho^M_r$ actions is an $R$-module.

Then $(\varphi, \psi, (\rho_r)_{r \in R})$ defines an additive functor

$$I : \mathcal{D} \longrightarrow \text{Mod}-R, \quad M \mapsto (\varphi(M)/\psi(M), (\rho^M_r)_{r \in R}).$$

We call any such functor an interpretation functor.
Let $R, S$ be rings.

**Theorem (Krause, Prest)**

Let $\mathcal{D}$ be a definable subcategory of Mod-$S$. An additive functor $I : \mathcal{D} \longrightarrow \text{Mod-}R$ is an interpretation functor if and only if $I$ commutes with direct limits and products.
Interpretation functors - Examples coming from algebra.

Let $R, S$ be rings. Let $_RB_S$ be an $R$-$S$-bimodule.

- If $B_S$ is finitely presented then $\text{Hom}_S(B, -) : \text{Mod}-S \to \text{Mod}-R$ is an interpretation functor.
- If $_RB$ is finitely presented then $- \otimes_R B : \text{Mod}-R \to \text{Mod}-S$ is an interpretation functor.
- If $B_S \in \text{FP}_2$ then $\text{Ext}_S(B, -) : \text{Mod}-S \to \text{Mod}-R$ is an interpretation functor.
- If $_RB \in \text{FP}_2$ then $\text{Tor}_R(B, -) : \text{Mod}-R \to \text{Mod}-S$ is an interpretation functor.

In particular, the equivalences coming from classical tilting are interpretation functors between definable subcategories.
Let $R, S$ be rings.

**Proposition**

Let $I : \text{Mod-}S \rightarrow \text{Mod-}R$ be an interpretation functor such that $\langle I \text{Mod-}S \rangle = \text{Mod-}R$. There is an $n \in \mathbb{N}$ and a lattice embedding $i : \text{pp}^1_R \hookrightarrow \text{pp}^n_S$.

**Reminder:** An $R$-module $M$ is **pure-injective** if any system of (inhomogeneous) linear equations over $R$, in arbitrary many variables, which is finitely solvable in $M$, has a solution in $M$.

**Remark**

Let $I : \text{Mod-}S \rightarrow \text{Mod-}R$ be an interpretation functor. If $M \in \text{Mod-}S$ is pure-injective then $IM$ is pure-injective.

**Theorem (G.)**

Let $I : \text{Mod-}S \rightarrow \text{Mod-}R$ be an interpretation functor such that $I$ maps finitely presented $S$-modules to finitely presented $R$-modules. If $I$ is full on finitely presented $S$-modules then $I$ is full on pure-injective $S$-modules.
Conjecture (Prest 80’s)

Let $A$ be a finite-dimensional $k$-algebra. If $A$ is of wild representation type then the theory of $A$-modules interprets the theory of $k\langle x, y \rangle$-modules.

Hence, if $k$ is countable, $A$ has undecidable theory of modules.

Conversely, if $A$ is tame then the theory of $A$-modules is decidable.
What does “theory of $A$-modules” mean?

A (first order) sentence in the language of $A$-modules is a statement, which can be assigned a truth value, built up from homogenous linear equations over $A$ in variables $\{x_i \mid i \in \mathbb{N}\}$, $\exists x_i$, $\forall x_i$, NOT, AND and OR.

Examples: Let $r, s \in A$.

$$\text{NOT}(\forall x_1 \exists x_2 \exists x_3 \ x_1 + x_2 \cdot r + x_3 \cdot s = 0)$$

is a sentence in the language of $A$-modules.

$$\forall x_1 \ \text{AND} \ x_2 \cdot r \ \text{and} \ x_1 + x_2 \cdot r = 0$$

are not sentences in the language of $A$-modules.

The theory of $A$-modules is the set of all sentences in the language of $A$-modules which are true in all $A$-modules.
From now on $k$ is an algebraically closed field.

**Definition**
A finite-dimensional $k$-algebra $\mathcal{A}$ is **wild** if:
there exists a **representation embedding**

$$F : \text{fin-}k\langle x, y \rangle \to \text{fin-}\mathcal{A}$$

i.e. $F$ is an exact $k$-linear functor which reflects isomorphism classes and sends indecomposable modules to indecomposable modules.

Equivalently, for every finite-dimensional $k$-algebra $B$ there exists a representation embedding

$$F : \text{fin-}B \to \text{fin-}\mathcal{A}.$$
Finite-dimensional algebras

Conjecture (Prest 80’s)

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Hence, if $k$ is countable, $A$ has undecidable theory of modules.

Conversely, if $A$ is tame then the theory of $A$-modules is decidable.
Definition
A finite-dimensional $k$-algebra $\mathcal{A}$ is **tame** if, for every dimension $d \in \mathbb{N}$, there is a finite number of $\mathcal{A}$-$k[x]$-bimodules $M_1, \ldots, M_{u(d)}$, which are finitely generated and free as $k[x]$-modules, such that almost all $d$-dimensional indecomposable $\mathcal{A}$-modules are of the form

$$M_i \otimes_{k[x]} k[x]/\langle x - \lambda \rangle$$

for some $1 \leq i \leq u(d)$ and some $\lambda \in k$.

Theorem (Drozd)

*Every finite-dimensional $k$-algebra is either tame or wild.*

Definition

Let $\mu(d)$ be the least possible value of $u(d)$ in the definition of a tame algebra. The finite-dimensional $k$-algebra $\mathcal{A}$ is **tame domestic** if $\mu(d)$ is bounded.
Theorem (G., Prest)

Let $\mathcal{A}, \mathcal{B}$ be finite-dimensional $k$-algebras. If $I : \text{Mod-}\mathcal{A} \to \text{Mod-}\mathcal{B}$ is a $k$-linear interpretation functor and $\langle I\text{Mod-}\mathcal{A} \rangle = \text{Mod-}\mathcal{B}$ then:

$\begin{align*}
\uparrow & \quad \text{if } \mathcal{A} \text{ is tame then } \mathcal{B} \text{ is tame} \\
\uparrow & \quad \text{if } \mathcal{A} \text{ is tame domestic then } \mathcal{B} \text{ is tame domestic} \\
\uparrow & \quad \text{if } \mathcal{A} \text{ is of polynomial growth then } \mathcal{B} \text{ is of polynomial growth} \\
\uparrow & \quad \text{if } \mathcal{A} \text{ is of non-exponential growth then } \mathcal{B} \text{ is of non-exponential growth}
\end{align*}$

Moreover, if $\mathcal{A}$ is wild then there exists a $k$-linear interpretation functor $I : \text{Mod-}\mathcal{A} \to \text{Mod-}\mathcal{B}$ such that $\langle I\text{Mod-}\mathcal{A} \rangle = \text{Mod-}\mathcal{B}$.

Corollary

A finite-dimensional $k$-algebra $\mathcal{A}$ is wild if and only if for every finite-dimensional $k$-algebra $\mathcal{B}$ there is a $k$-linear interpretation functor $I : \text{Mod-}\mathcal{A} \to \text{Mod-}\mathcal{B}$ such that $\langle I\text{Mod-}\mathcal{A} \rangle = \text{Mod-}\mathcal{B}$.
Wild implies undecidability

A finite dimensional $k$-algebra $A$ is **finitely controlled wild** if there is a representation embedding

$$F : \text{fin-}k\langle x, y \rangle \longrightarrow \text{fin-}A$$

and $C \in \text{fin-}k\langle x, y \rangle$ such that for all $N, M \in \text{fin-}k\langle x, y \rangle$

$$\text{Hom}_B(FM, FN) = F\text{Hom}(M, N) \oplus \text{Hom}(FM, FN)_C$$

where $\text{Hom}(FM, FN)_C$ is the set of maps which factor through some $C^n$.

**Theorem (G., Prest)**

Let $A$ be a finite-dimensional $k$-algebra. If $A$ is finitely controlled wild then there is a $k$-linear essentially surjective interpretation functor $I : \text{Mod-}A \rightarrow \text{Mod-}k\mathbb{K}_3$.

**Corollary**

In the above situation, the theory of $A$-modules interprets the theory of $k\mathbb{K}_3$-modules. In particular, if $k$ is countable, the theory of $A$-Mod is undecidable.