AC-Gorenstein homological algebra and AC-Gorenstein rings

James Gillespie

Representation Theory and Related Topic Seminar at Northeastern: December 1, 2017
This talk is mainly based on the papers:


1. Absolutely clean (AC) and level modules - Duality

2. Injective (resp. projective) abelian model structures

3. Gorenstein AC-injective (resp. AC-projective) modules

4. AC-Gorenstein rings
Main Idea: Different notions of “finite” module leads to different notions of injectivity and flatness.

Definition: An $R$-module $M$ is said to be of type $FP_\infty$ if it has a projective resolution

$$\cdots \to P_2 \to P_1 \to P_0 \to M \to 0$$

where each $P_i$ is finitely generated.

Note: $M$ of type $FP_\infty \implies M$ f.p. $\implies M$ f.g. But...
Let $R$ be a ring. The following are equivalent:

1. $R$ is (left) Noetherian.

2. All finitely generated (left) modules are of type $FP_\infty$.

3. The class of finitely generated (left) modules is thick. That is, it is closed under retracts and satisfies the 2 out of 3 property for short exact sequences.
Injective modules over Noetherian rings

This is the REASON behind the following well-known Fact:

**Fact:** The following are equivalent.

1. $R$ is (left) Noetherian.
2. The class of injective (left) $R$-modules is closed under direct limits (or just direct sums).

**(Proof of 1. implies 2.)** For any ring, Baer’s criterion implies that $I$ is injective iff $\text{Ext}_R^1(M, I) = 0$ for any finitely generated $M$. If $R$ is Noetherian and $M$ is finitely generated, take a projective resolution $P_* \to M$ with each $P_n$ finitely generated (so f.p). Then...

$$\text{Ext}_R^1(M, \lim_{\alpha} I_\alpha) = H^1[\text{Hom}_R(P_*, \lim_{\alpha} I_\alpha)] \cong H^1[\lim_{\alpha} \text{Hom}_R(P_*, I_\alpha)]$$

$$\cong \lim_{\alpha} H^1\text{Hom}_R(P_*, I_\alpha) \cong \lim_{\alpha} \text{Ext}_R^1(M, I_\alpha) = 0$$
Let $R$ be a ring. The following are equivalent:

1. $R$ is (left) coherent.
2. All finitely presented (left) modules are of type $FP_\infty$.
3. The class of finitely presented (left) modules is thick.
Absolutely pure and flat modules over coherent rings

This is the REASON behind the following Fact:

**Fact:** The following are equivalent.

1. $R$ is (left) coherent.
2. The class of absolutely pure (left) $R$-modules is closed under direct limits.

(Reason 1. implies 2.) Absolutely pure means $\text{Ext}^1_R(M, A) = 0$ for all f.p. $M$. So...

\[
\text{Ext}^1_R(M, \lim A_\alpha) = H^1[\text{Hom}_R(P_*, \lim A_\alpha)] \cong H^1[\lim \text{Hom}_R(P_*, A_\alpha)]
\]

\[
\cong \lim H^1 \text{Hom}_R(P_*, A_\alpha) \cong \lim \text{Ext}^1_R(M, A_\alpha) = 0
\]
**Fact:** Bieri (Geometric Group Theory) showed that the class of $FP_\infty$-modules is thick over ANY ring $R$. This suggests...

**Definition:** An $R$-module $N$ is called **absolutely clean** (or $FP_\infty$-injective) if $\text{Ext}^1_R(M, N) = 0$ for every $M$ of type $FP_\infty$.

We call a s.e.s. $0 \rightarrow N \rightarrow A \rightarrow B \rightarrow 0$ “clean” if it remains exact after applying $\text{Hom}_R(M, -)$ for any $M$ of type $FP_\infty$. So...

**Fact:** A module $N$ is absolutely clean if and only if every short exact sequence $0 \rightarrow N \rightarrow A \rightarrow B \rightarrow 0$ is clean.
Absolutely clean modules over coherent and Noetherian rings

Recall: $M$ of type $FP_\infty \implies M$ f.p. $\implies M$ f.g.

So: $I$ injective $\implies I$ absolutely pure $\implies I$ absolutely clean.

But...

**Fact:** (1) $R$ is (left) coherent $\iff$ The absolutely clean (left) modules coincide with the absolutely pure modules.

(2) $R$ is (left) Noetherian $\iff$ The absolutely clean (left) modules coincide with the injective modules.
**Proposition:** Absolutely clean modules have the following properties over ANY ring $R$.

- The class of absolutely clean modules is closed under pure submodules and pure quotients.
- The class of absolutely clean modules is coresolving; that is, it contains the injectives and is closed under extensions and cokernels of monomorphisms.
- The class of absolutely clean modules is closed under products, sums, retracts, direct limits, and transfinite extensions.
- There is some regular cardinal $\kappa$ such that every absolutely clean module is a transfinite extension of absolutely clean modules with cardinality bounded by $\kappa$. 

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All of what we have done has a “dual”.

Recall...

**Chase’s Theorem:** The following are equivalent.

1. $R$ is (right) coherent.
2. The class of flat (left) $R$-modules is closed under products.

*(Reason 1. implies 2.)* Because... For any ring $R$ we have $F$ is flat iff $\text{Tor}^R_1(M, F) = 0$ for all finitely presented $M$. So, if $R$ is coherent,

$$\text{Tor}^R_1(M, \prod F_\alpha) = H_1(P_* \otimes_R \prod F_\alpha) \cong H_1(\prod (P_* \otimes_R F_\alpha))$$

$$\cong \prod H_1(P_* \otimes_R F_\alpha) = \prod \text{Tor}^R_1(M, F_\alpha) = 0$$
This suggests the following generalization of flat modules:

**Definition:** A (left) $R$-module $N$ is called **level** if $\text{Tor}_1^R(M, N) = 0$ for every (right) $R$-module $M$ of type $FP_{\infty}$.

Note: $N$ flat $\implies N$ level.

But...

**Prop:** $R$ is (right) coherent $\iff$ the level (left) modules coincide with the flat modules.

So... Level = Flat for (right) Noetherian rings too.
Properties of level modules

**Proposition:** Level modules have the following properties over ANY ring $R$.

- The class of level modules is closed under pure submodules and pure quotients.
- The class of level modules is resolving; that is, it contains the projectives and is closed under extensions and kernels of epimorphisms.
- The class of level modules is closed under *products*, sums, retracts, direct limits, and transfinite extensions.
- There is some regular cardinal $\kappa$ such that every level module is a transfinite extension of level modules with cardinality bounded by $\kappa$. 
Theorem: Let $R$ be ANY ring.

- (Level, Absolutely clean) form a “complete duality pair” with respect to taking character modules $M^+ = \text{Hom}_\mathbb{Z}(M, \mathbb{Q}/\mathbb{Z})$.

\[ L \text{ level} \iff L^+ \text{ absolutely clean} \]

\[ A \text{ absolutely clean} \iff A^+ \text{ level} \]

- Every module has a level cover and the level modules form the left half of a complete hereditary cotorsion pair.
Outline

1. Absolutely clean (AC) and level modules - Duality
2. Injective (resp. projective) abelian model structures
3. Gorenstein AC-injective (resp. AC-projective) modules
4. AC-Gorenstein rings
Definition of Complete Cotorsion Pair.

Let $\mathcal{A}$ be an abelian category, such as $R$-Mod or $\text{Ch}(R)$.

A pair of classes $(\mathcal{X}, \mathcal{Y})$ of objects in $\mathcal{A}$ is a cotorsion pair if the following conditions hold:

- $X \in \mathcal{X}$ iff $\text{Ext}^1_{\mathcal{A}}(X, Y) = 0$ for all $Y \in \mathcal{Y}$.
- $Y \in \mathcal{Y}$ iff $\text{Ext}^1_{\mathcal{A}}(X, Y) = 0$ for all $X \in \mathcal{X}$.

We say the cotorsion pair is complete if for any $A \in \mathcal{A}$ there exist short exact sequences $Y \rightarrowtail X \twoheadrightarrow A$ with $X \in \mathcal{X}$ and $Y \in \mathcal{Y}$, and $A \rightarrowtail Y' \twoheadrightarrow X'$ with $X' \in \mathcal{X}$ and $Y' \in \mathcal{Y}$.

We say the cotorsion pair is hereditary if each of the following hold:

1. For any s.e.s. $A \rightarrowtail X' \twoheadrightarrow X$, if $X, X' \in \mathcal{X}$ then also $A \in \mathcal{X}$.
2. For any s.e.s. $Y \rightarrowtail Y' \twoheadrightarrow B$, if $Y, Y' \in \mathcal{Y}$ then also $B \in \mathcal{Y}$.
**IDEA:** The theory of abelian model categories provides a powerful method for constructing triangulated categories. Through Hovey’s correspondence, there is a bijective correspondence between “injective cotorsion pairs” on $\mathcal{A}$ and “injective abelian model structures” on $\mathcal{A}$.

**Definition:** If $\mathcal{A}$ has enough injectives, then we call a cotorsion pair $(\mathcal{W}, \mathcal{F})$ an **injective cotorsion pair** if it is complete, $\mathcal{W}$ is **thick**, and $\mathcal{W} \cap \mathcal{F}$ coincides with the class of injective objects.

The objects in $\mathcal{F}$ are then called **fibrant** objects.

If $\mathcal{A}$ has enough projectives, there is also the dual notion of a **projective cotorsion pair** $(\mathcal{C}, \mathcal{V})$ with **cofibrant** objects $\mathcal{C}$.
What is the point?

FACTS: Assume \( \mathcal{M} = (\mathcal{W}, \mathcal{F}) \) is an injective cotorsion pair on an abelian category \( \mathcal{A} \). The following fundamental facts hold:

- There is an associated **homotopy category**, \( \text{Ho}(\mathcal{M}) \).
  1. Objects in \( \text{Ho}(\mathcal{M}) \) are the same as the objects in \( \mathcal{A} \).
  2. \( \text{Hom}_{\text{Ho}(\mathcal{M})}(X, Y) = \text{Hom}_{\mathcal{A}}(FX, FY)/\sim \) where \( f \sim g \) iff \( g - f \) factors through an injective object.

- There is a canonical functor \( \gamma: \mathcal{A} \to \text{Ho}(\mathcal{M}) \) which is a “triangulated localization” with respect to the trivial objects \( \mathcal{W} \). That is, \( \text{Ho}(\mathcal{M}) \) is triangulated, the functor sends short exact sequences to exact triangles, and \( \gamma \) is universally initial among such functors taking \( \mathcal{W} \) to 0.

- The full subcategory \( \mathcal{F} \) is a Frobenius category and there is a triangulated equivalence of categories: \( \text{Ho}(\mathcal{M}) \cong \mathcal{F}/\sim \).
Examples of Injective Cotorsion Pairs in Ch($R$)

The injective model structure

$$(\mathcal{E}, dg\tilde{I}) = (\text{Exact complexes, DG-Injective complexes}) \text{ in Ch}(R).$$

$${ \text{Ho}[\text{Ch}(R)] = D(R) \cong dg\tilde{I}/\sim}$$

The Inj model structure

$$(\mathcal{W}_1, dw\tilde{I}) = (\perp dw\tilde{I}, \text{Complexes of injectives}) \text{ in Ch}(R).$$

$${ \text{Ho}[\text{Ch}(R)] \cong K(Inj) = dw\tilde{I}/\sim}$$

The exact Inj model structure

$$(\mathcal{W}_2, ex\tilde{I}) = (\perp ex\tilde{I}, \text{Exact complexes of injectives}) \text{ in Ch}(R).$$

$${ \text{Ho}[\text{Ch}(R)] \cong K_{ex}(Inj) = ex\tilde{I}/\sim}$$
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Recall:

**Definition**: An $R$-module $M$ is called **Gorenstein injective** if there exists an exact complex of injectives

$$
\cdots \rightarrow I_1 \rightarrow I_0 \rightarrow I^0 \rightarrow I^1 \rightarrow \cdots
$$

with $M = \ker (I^0 \rightarrow I^1)$ and which remains exact after applying $\text{Hom}_R(J, -)$ for any injective module $J$. 
Recall:

**Definition:** An $R$-module $M$ is called **Gorenstein AC-injective** if there exists an exact complex of injectives

$$
\cdots \to I_1 \to I_0 \to I^0 \to I^1 \to \cdots
$$

with $M = \ker (I^0 \to I^1)$ and which remains exact after applying $\text{Hom}_R(J, -)$ for any **absolutely clean module** $J$. 

Noetherian Case: $R$ Noetherian $\implies$ The Gorenstein AC-injective modules coincide with the usual Gorenstein injective modules.

(Reason) $R$ Noetherian implies that the absolutely clean modules coincide with the injective modules.

Coherent Case: $R$ coherent $\implies$ The Gorenstein AC-injective modules coincide with modules I call “Ding injective” modules.

(Reason) $R$ coherent implies that the absolutely clean modules coincide with the absolutely pure modules.
Theorem (Bravo/Gillespie/Hovey)

Let $R$ be any ring and let $\mathcal{GI}$ denote the class of Gorenstein AC-injective modules. Set $\mathcal{W}_{inj} = \perp \mathcal{GI}$. Then $\mathcal{M}_{inj} = (\mathcal{W}_{inj}, \mathcal{GI})$ is an injective cotorsion pair. That is,

- $\mathcal{W}_{inj}$ is thick, and contains all projectives and injectives.
- $\mathcal{W}_{inj} \cap \mathcal{GI}$ is the class of injectives.
- $(\mathcal{W}_{inj}, \mathcal{GI})$ is complete, in fact, cogenerated by a set.

Proof.

Explain how we built it on chain complexes...
Corollary: The “Gorenstein AC-injective model” on $R$-Mod.

$\mathcal{M}_{inj}$ is an injective abelian model structure on $R$-Mod whose fibrant objects are the Gorenstein AC-injectives. We get an equivalence of triangulated categories:

$$\text{Ho}[\mathcal{M}_{inj}] \simeq GI/\sim$$

Corollary: “Gorenstein AC-injective pre-envelopes” of $R$-modules.

Every $R$-module has a Gorenstein AC-injective pre-envelope. (These are fibrant replacements.)
Recall:

**Definition**: An $R$-module $M$ is called **Gorenstein projective** if there exists an exact complex of projectives

$$
\cdots \to P_1 \to P_0 \to P^0 \to P^1 \to \cdots
$$

with $M = \ker (P^0 \to P^1)$ and which remains exact after applying $\text{Hom}_R(-, Q)$ for any projective module $Q$. 
Recall:

**Definition:** An $R$-module $M$ is called **Gorenstein AC-projective** if there exists an exact complex of projectives

$$
\cdots \to P_1 \to P_0 \to P^0 \to P^1 \to \cdots
$$

with $M = \ker (P^0 \to P^1)$ and which remains exact after applying $\text{Hom}_R (-, Q)$ for any level module $Q$. 
Let $R$ be any ring and let $\mathcal{GP}$ denote the class of Gorenstein AC-projective modules. Set $\mathcal{W}_{\text{prj}} = \mathcal{GP}^\perp$. Then $\mathcal{M}_{\text{prj}} = (\mathcal{GP}, \mathcal{W}_{\text{prj}})$ is a projective cotorsion pair. That is,

- $\mathcal{W}_{\text{prj}}$ is thick, and contains all projectives and injectives.
- $\mathcal{GP} \cap \mathcal{W}_{\text{prj}}$ is the class of projectives.
- $(\mathcal{GP}, \mathcal{W}_{\text{prj}})$ is complete, in fact, cogenerated by a set.
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Summary:

- For an arbitrary ring $R$, we have two abelian model structures on $R$-Mod: $\mathcal{M}_{inj} = (\mathcal{W}_{inj}, \mathcal{GI})$ and $\mathcal{M}_{prj} = (\mathcal{GP}, \mathcal{W}_{prj})$.
- $\mathcal{W}_{inj}$ contains all projective and injective modules, in fact, it contains all abs. clean modules.
- $\mathcal{W}_{prj}$ contains all projective and injective modules, in fact, it contains all level modules.

Open Question: When do we have $\mathcal{W}_{inj} = \mathcal{W}_{prj}$?

Motivation: This would assign to $R$, a unique stable module cat:

$$\text{Stmod}(R) := \text{Ho}[\mathcal{M}_{inj}] = \text{Ho}[\mathcal{M}_{prj}]$$
Recall:

**Definition:** A ring $R$ is a **Gorenstein ring** if $R$ is Noetherian and there is an upper bound, say $n$, on the injective dimension of all flat modules.

**Theorem (Fundamental Theorem)**

Then the following are equivalent for any $R$-module $M$:

1. $fd(M) < \infty$.
2. $fd(M) \leq n$.
3. $id(M) < \infty$.
4. $id(M) \leq n$. 
Recall:

**Definition:** A ring $R$ is a **AC-Gorenstein ring** if $R$ is any ring and there is an upper bound, say $n$, on the abs. clean dimension of all level modules.

**Theorem (Fundamental Theorem)**

*Then the following are equivalent for any $R$-module $M$:*

1. $\text{ld}(M) < \infty$.
2. $\text{ld}(M) \leq n$.
3. $\text{ad}(M) < \infty$.
4. $\text{ad}(M) \leq n$. 
Noetherian and coherent cases of AC-Gorenstein rings

**Noetherian Case:** A Noetherian AC-Gorenstein ring is precisely a Iwanaga-Gorenstein ring.

(Reason) level = flat, and, absolutely clean = injective.

**Coherent Case:** A coherent AC-Gorenstein ring is precisely a Ding-Chen ring.

(Reason) level = flat, and, absolutely clean = absolutely pure.
Particular examples of AC-Gorenstein rings

Example
If $R$ is commutative coherent and absolutely pure (as a module over itself), then the group ring $RG$ is an AC-Gorenstein ring for any locally finite group $G$.

Example
If $R$ is commutative coherent and of finite absolutely pure dimension, then the group ring $RG$ is an AC-Gorenstein ring for any finite group $G$.

Ques: When is a group algebra $RG$ an AC-Gorenstein ring?

Examples
Let $R$ be AC-Gorenstein. Then the graded ring $A := R[x]/(x^2)$ is also AC-Gorenstein. In some cases, we have $\text{Stmod}(A) \cong \mathcal{D}(R)$.
Main theorem on AC-Gorenstein rings

**Theorem**

Let $R$ be an AC-Gorenstein ring and let $\mathcal{W}$ denote the class of all module of finite level (equivalently, absolutely clean) dimension. Then:

- $\mathcal{W}_{\text{inj}} = \mathcal{W} = \mathcal{W}_{\text{prj}}$
- $(\mathcal{GP}, \mathcal{W})$ and $(\mathcal{W}, \mathcal{GI})$ are “balanced”.
- $\text{Stmod}(R)(:= \text{Ho}[\mathcal{M}_{\text{inj}}] = \text{Ho}[\mathcal{M}_{\text{prj}}])$ is a compactly generated triangulated category.
Thank You!