Algebraic models and the Tangent Cone theorem

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RESONANCE VARIETIES OF A CDGA

- Let $A = (A^\bullet, \mathrm{d})$ be a commutative, differential graded $\mathbb{C}$-algebra.
  - Multiplication $\cdot \colon A^i \otimes A^j \to A^{i+j}$ is graded-commutative.
  - Differential $\mathrm{d} \colon A^i \to A^{i+1}$ satisfies the graded Leibniz rule.

- Assume
  - A is connected, i.e., $A^0 = \mathbb{C}$.
  - A is of finite-type, i.e., $\dim A^i < \infty$ for all $i \geqslant 0$.

- For each $a \in Z^1(A) \cong H^1(A)$, we get a cochain complex,

$$(A^\bullet, \delta_a)\colon \ A^0 \xrightarrow{\delta_a^0} A^1 \xrightarrow{\delta_a^1} A^2 \xrightarrow{\delta_a^2} \cdots,$$

with differentials $\delta_a^i(u) = a \cdot u + \mathrm{d}\, u$, for all $u \in A^i$.

- *Resonance varieties*:

$$\mathcal{R}^i(A) = \{a \in H^1(A) \mid H^i(A^\bullet, \delta_a) \neq 0\}.$$

- Fix $\mathbb{C}$-basis $\{e_1, \dots, e_n\}$ for $H^1(A)$, and let $\{x_1, \dots, x_n\}$ be dual basis for $H_1(A) = H^1(A)^\vee$.
- Identify $\mathrm{Sym}(H_1(A))$ with $S = \mathbb{C}[x_1, \dots, x_n]$, the coordinate ring of the affine space $H^1(A)$.
- Define a cochain complex of free S-modules,

$$(A^\bullet \otimes S, \delta)\colon \quad \cdots \longrightarrow A^i \otimes S \xrightarrow{\delta^i} A^{i+1} \otimes S \xrightarrow{\delta^{i+1}} A^{i+2} \otimes S \longrightarrow \cdots,$$

where $\delta^i(u \otimes s) = \sum_{j=1}^n e_j u \otimes s x_j + \mathrm{d}\, u \otimes s$.

- The specialization of $A \otimes S$ at $a \in H^1(A)$ coincides with (A, δ_a).
- The cohomology support loci $\widetilde{\mathcal{R}}^i(A) = \mathrm{supp}(H^i(A^\bullet \otimes S, \delta))$ are subvarieties of $H_1(A)$.

- Let $(A_\bullet \otimes S, \partial)$ be the dual chain complex.
- The homology support loci $\widetilde{\mathcal{R}}_i(A) = \operatorname{supp}(H_i(A_\bullet \otimes S, \partial))$ are subvarieties of $H_1(A)$.
- Using a result of [Papadima–S. 2014], we obtain:

Theorem

For each $q \geqslant 0$, the duality isomorphism $H^1(A) \cong H_1(A)$ restricts to an isomorphism $\bigcup_{i \leqslant q} \mathcal{R}^i(A) \cong \bigcup_{i \leqslant q} \widetilde{\mathcal{R}}_i(A)$.

- We also have $\mathcal{R}^i(A) \cong \mathcal{R}_i(A)$.
- In general, though, $\widetilde{\mathcal{R}}^i(A) \not\cong \widetilde{\mathcal{R}}_i(A)$.
- If $d = 0$, then all the resonance varieties of A are homogeneous.
- In general, though, they are not.

EXAMPLE

- Let A be the exterior algebra on generators a, b in degree 1, endowed with the differential given by $\mathrm{d}\, a = 0$ and $\mathrm{d}\, b = b \cdot a$.
- $H^1(A) = \mathbb{C}$, generated by a. Set $S = \mathbb{C}[x]$. Then:

$$A_\bullet \otimes S\colon\ \ S \xrightarrow{\partial_2 = \left(\begin{smallmatrix} 0 \\ x-1 \end{smallmatrix}\right)} S^2 \xrightarrow{\partial_1 = \left(\begin{smallmatrix} x & 0 \end{smallmatrix}\right)} S\ .$$

- Hence, $H_1(A_\bullet \otimes S) = S/(x-1)$, and so $\widetilde{\mathcal{R}}_1(A) = \{1\}$. Using the above theorem, we conclude that $\mathcal{R}^1(A) = \{0, 1\}$.
- $\mathcal{R}^1(A)$ is a non-homogeneous subvariety of $\mathbb{C}$.
- $H^1(A_\bullet \otimes S) = S/(x)$, and so $\widetilde{\mathcal{R}}^1(A) = \{0\} \neq \widetilde{\mathcal{R}}_1(A)$.

RESONANCE VARIETIES OF A SPACE

- Let X be a connected, finite-type CW-complex.
- We may take $A = H^*(X, \mathbb{C})$ with $d = 0$, and get the usual resonance varieties, $\mathcal{R}^i(X) := \mathcal{R}^i(A)$.
- Or, we may take (A, d) to be a finite-type cdga, weakly equivalent to Sullivan's model $A_{\mathrm{PL}}(X)$, if such a cdga exists.
- If X is *formal*, then $(H^*(X, \mathbb{C}), d = 0)$ is such a finite-type model.
- Finite-type cdga models exist even for possibly non-formal spaces, such as nilmanifolds and solvmanifolds, Sasakian manifolds, smooth quasi-projective varieties, etc.

THEOREM (MACINIC, PAPADIMA, POPESCU, S. – 2013)

Suppose there is a finite-type CDGA (A, d) such that $A_{\mathrm{PL}}(X) \simeq A$. Then, for each $i \geqslant 0$, the tangent cone at 0 to the resonance variety $\mathcal{R}^i(A)$ is contained in $\mathcal{R}^i(X)$.

In general, we cannot replace $\mathrm{TC}_0(\mathcal{R}^i(A))$ by $\mathcal{R}^i(A)$.

EXAMPLE

- Let $X = S^1$, and take $A = \bigwedge(a, b)$ with $\mathrm{d}\, a = 0$, $\mathrm{d}\, b = b \cdot a$.
- Then $\mathcal{R}^1(A) = \{0, 1\}$ is not contained in $\mathcal{R}^1(X) = \{0\}$, though $\mathrm{TC}_0(\mathcal{R}^1(A)) = \{0\}$ is.

- A rationally defined CDGA (A, d) has *positive weights* if each A^i can be decomposed into weighted pieces A^i_α, with positive weights in degree 1, and in a manner compatible with the CDGA structure:
  1. $A^i = \bigoplus_{\alpha\in\mathbb{Z}} A^i_\alpha$.
  2. $A^1_\alpha = 0$, for all $\alpha \leqslant 0$.
  3. If $a \in A^i_\alpha$ and $b \in A^j_\beta$, then $ab \in A^{i+j}_{\alpha+\beta}$ and $\mathrm{d}\, a \in A^{i+1}_\alpha$.
- A space X is said to have positive weights if $A_{\mathrm{PL}}(X)$ does.
- If X is formal, then X has positive weights, but not conversely.

THEOREM (DIMCA–PAPADIMA 2014, MPPS)

Suppose there is a rationally defined, finite-type CDGA (A, d) *with positive weights, and a* q*-equivalence between* $A_{\mathrm{PL}}(X)$ *and* A *preserving* $\mathbb{Q}$*-structures. Then, for each* $i \leqslant q$*,*

1. $\mathcal{R}^i(A)$ *is a finite union of rationally defined linear subspaces of* $H^1(A)$.
2. $\mathcal{R}^i(A) \subseteq \mathcal{R}^i(X)$.

EXAMPLE

- Let X be the 3-dimensional Heisenberg nilmanifold.
- All cup products of degree 1 classes vanish; thus, $\mathcal{R}^1(X) = H^1(X, \mathbb{C}) = \mathbb{C}^2$.
- Model $A = \bigwedge(a, b, c)$ generated in degree 1, with $\mathrm{d}\, a = \mathrm{d}\, b = 0$ and $\mathrm{d}\, c = a \cdot b$.
- This is a finite-dimensional model, with positive weights: $\mathrm{wt}(a) = \mathrm{wt}(b) = 1$, $\mathrm{wt}(c) = 2$.
- Writing $S = \mathbb{C}[x, y]$, we get

$$A_\bullet \otimes S\colon \ \cdots \longrightarrow S^3 \xrightarrow{\begin{pmatrix} y & 0 & 0 \\ -x & 0 & 0 \\ 1 & -x & -y \end{pmatrix}} S^3 \xrightarrow{\begin{pmatrix} x & y & 0 \end{pmatrix}} S \,.$$

- Hence $H_1(A_\bullet \otimes S) = S/(x, y)$, and so $\mathcal{R}^1(A) = \{0\}$.

CHARACTERISTIC VARIETIES

- Let X be a finite-type, connected CW-complex.
  - $\pi = \pi_1(X, x_0)$: a finitely generated group.
  - $\mathrm{Char}(X) = \mathrm{Hom}(\pi, \mathbb{C}^*)$: an abelian, algebraic group.
  - $\mathrm{Char}(X)^0 \cong (\mathbb{C}^*)^n$, where $n = b_1(X)$.
- Characteristic varieties of X:

$$\mathcal{V}^i(X) = \{\rho \in \mathrm{Char}(X) \mid H^i(X, \mathbb{C}_\rho) \neq 0\}.$$

THEOREM (LIBGOBER 2002, DIMCA–PAPADIMA–S. 2009)

$$\tau_1(\mathcal{V}^i(X)) \subseteq \mathrm{TC}_1(\mathcal{V}^i(X)) \subseteq \mathcal{R}^i(X)$$

- Here, if $W \subset (\mathbb{C}^*)^n$ is an algebraic subset, then

$$\tau_1(W) := \{z \in \mathbb{C}^n \mid \exp(\lambda z) \in W, \text{ for all } \lambda \in \mathbb{C}\}.$$

- This is a finite union of rationally defined linear subspaces of $\mathbb{C}^n$.

THEOREM (DIMCA–PAPADIMA 2014)

Suppose $A_{\mathrm{PL}}(X)$ is q-equivalent to a finite-type model (A, d). Then $\mathcal{V}^i(X)_{(1)} \cong \mathcal{R}^i(A)_{(0)}$, for all $i \leqslant q$.

COROLLARY

If X is a q-formal space, then $\mathcal{V}^i(X)_{(1)} \cong \mathcal{R}^i(X)_{(0)}$, for all $i \leqslant q$.

- A precursor to corollary can be found in work of Green–Lazarsfeld on the cohomology jump loci of compact Kähler manifolds.
- The case when $q = 1$ was first established in [DPS-2009].
- Further developments in work of Budur–Wang [2013].

THE TANGENT CONE THEOREM

THEOREM

Suppose $A_{\mathrm{PL}}(X)$ is q-equivalent to a finite-type CDGA A. Then, $\forall i \leqslant q$,

1. $\mathrm{TC}_1(\mathcal{V}^i(X)) = \mathrm{TC}_0(\mathcal{R}^i(A))$.
2. *If, moreover, A has positive weights, and the q-equivalence $A_{\mathrm{PL}}(X) \simeq A$ preserves $\mathbb{Q}$-structures, then $\mathrm{TC}_1(\mathcal{V}^i(X)) = \mathcal{R}^i(A)$.*

THEOREM (DPS-2009, DP-2014)

Suppose X is a q-formal space. Then, for all $i \leqslant q$,

$$\tau_1(\mathcal{V}^i(X)) = \mathrm{TC}_1(\mathcal{V}^i(X)) = \mathcal{R}^i(X).$$

COROLLARY

If X is q-formal, then, for all $i \leqslant q$,

1. *All irreducible components of $\mathcal{R}^i(X)$ are rationally defined subspaces of $H^1(X, \mathbb{C})$.*
2. *All irreducible components of $\mathcal{V}^i(X)$ which pass through the origin are algebraic subtori of $\mathrm{Char}(X)^0$, of the form $\exp(L)$, where L runs through the linear subspaces comprising $\mathcal{R}^i(X)$.*

The Tangent Cone theorem can be used to detect non-formality.

EXAMPLE

- Let $\pi = \langle x_1, x_2 \mid [x_1, [x_1, x_2]] \rangle$.
- Then $\mathcal{V}^1(\pi) = \{t_1 = 1\}$, and so $\tau_1(\mathcal{V}^1(\pi)) = \mathrm{TC}_1(\mathcal{V}^1(\pi)) = \{x_1 = 0\}$.
- On the other hand, $\mathcal{R}^1(\pi) = \mathbb{C}^2$, and so π is not 1-formal.

EXAMPLE (DPS 2009)

Let $\pi = \langle x_1, \ldots, x_4 \mid [x_1, x_2], [x_1, x_4][x_2^{-2}, x_3], [x_1^{-1}, x_3][x_2, x_4] \rangle$. Then $\mathcal{R}^1(\pi) = \{z \in \mathbb{C}^4 \mid z_1^2 - 2z_2^2 = 0\}$: a quadric which splits into two linear subspaces over $\mathbb{R}$, but is irreducible over $\mathbb{Q}$. Thus, π is not 1-formal.

EXAMPLE (S.–YANG–ZHAO 2015)

Let π be a finitely presented group with $\pi_{\mathrm{ab}} = \mathbb{Z}^3$ and

$$\mathcal{V}^1(\pi) = \{(t_1, t_2, t_3) \in (\mathbb{C}^*)^3 \mid (t_2 - 1) = (t_1 + 1)(t_3 - 1)\},$$

This is a complex, 2-dimensional torus passing through the origin, but this torus does not embed as an algebraic subgroup in $(\mathbb{C}^*)^3$. Indeed,

$$\tau_1(\mathcal{V}^1(\pi)) = \{x_2 = x_3 = 0\} \cup \{x_1 - x_3 = x_2 - 2x_3 = 0\}.$$

Hence, π is not 1-formal.

GYSIN MODELS

- Let X be a (connected) smooth quasi-projective variety.
- Let $\overline{X}$ be a "good" compactification, i.e., $X = \overline{X} \setminus D$, for some normal-crossings divisor $D = \{D_1, \dots, D_m\}$.
- Algebraic model: $A = A(\overline{X}, D)$ (Morgan's *Gysin model*): a rationally defined, bigraded CDGA, with $A^i = \bigoplus_{p+q=i} A^{p,q}$ and

$$A^{p,q} = \bigoplus_{|S|=q} H^p\Big(\bigcap_{k\in S} D_k, \mathbb{C}\Big)(-q)$$

- Multiplication $A^{p,q} \cdot A^{p',q'} \subseteq A^{p+p',q+q'}$ from cup-product in $\overline{X}$.
- Differential $\mathrm{d}\colon A^{p,q} \to A^{p+2,q-1}$ from intersections of divisors.
- Model has positive weights: $\mathrm{wt}(A^{p,q}) = p + 2q$.
- Improved version by Dupont [2013]: divisor D is allowed to have "arangement-like" singularities.

- Suppose $X = \Sigma$ is a connected, smooth algebraic curve.
- Then Σ admits a canonical compactification, $\overline{\Sigma}$, and thus, a canonical Gysin model, $A(\Sigma)$.

EXAMPLE

Let $\Sigma = E^*$ be a once-punctured elliptic curve. Then $\overline{\Sigma} = E$, and

$$A(\Sigma) = \bigwedge(a, b, e)/(ae, be)$$

where a, b are in bidegree $(1, 0)$ and e in bidegree $(0, 1)$, while $\mathrm{d}\, a = \mathrm{d}\, b = 0$ and $\mathrm{d}\, e = ab$.

THE TANGENT CONE THEOREM

THEOREM (BUDUR, WANG 2013)

Let X be a smooth quasi-projective variety. Then each characteristic variety $\mathcal{V}^i(X)$ is a finite union of torsion-translated subtori of $\mathrm{Char}(X)$.

THEOREM

Let $A(X)$ be a Gysin model for X. Then, for each $i \geqslant 0$,

$$\tau_1(\mathcal{V}^i(X)) = \mathrm{TC}_1(\mathcal{V}^i(X)) = \mathcal{R}^i(A(X)) \subseteq \mathcal{R}^i(X).$$

Moreover, if X is q-formal, the last inclusion is an equality, for all $i \leqslant q$.

EXAMPLE

Let X be the $\mathbb{C}^*$-bundle over $E = S^1 \times S^1$ with $e = 1$. Then $\mathcal{V}^1(X) = \{1\}$, and so $\tau_1(\mathcal{V}^1(X)) = \mathrm{TC}_1(\mathcal{V}^1(X)) = \{0\}$. On the other hand, $\mathcal{R}^1(X) = \mathbb{C}^2$, and so X is not 1-formal.

A holomorphic map $f\colon X \to \Sigma$ is *admissible* if f is surjective, has connected generic fiber, and the target Σ is a connected, smooth complex curve with $\chi(X) < 0$.

THEOREM (ARAPURA 1997)

The map $f \mapsto f^*(\mathrm{Char}(\Sigma))$ yields a bijection between the set $\mathcal{E}_X$ of equivalence classes of admissible maps $X \to \Sigma$ and the set of positive-dimensional, irreducible components of $\mathcal{V}^1(X)$ containing 1.

THEOREM (DP 2014, MPPS 2013)

$$\mathcal{R}^1(A(X)) = \bigcup_{f\in\mathcal{E}_X} f^*(H^1(A(\Sigma))).$$

THEOREM (DPS 2009)

Suppose X is 1-formal. Then $\mathcal{R}^1(X) = \bigcup_{f\in\mathcal{E}_X} f^*(H^1(\Sigma, \mathbb{C}))$. Moreover, all the linear subspaces in this decomposition have dimension $\geqslant 2$, and any two distinct ones intersect only at 0.

APPLICATIONS OF COHOMOLOGY JUMP LOCI

- Homological and geometric finiteness of regular abelian covers
  - Bieri–Neumann–Strebel–Renz invariants
  - Dwyer–Fried invariants
- Obstructions to (quasi-) projectivity
  - Right-angled Artin groups and Bestvina–Brady groups
  - 3-manifold groups, Kähler groups, and quasi-projective groups
- Resonance varieties and representations of Lie algebras
  - Homological finiteness in the Johnson filtration of automorphism groups
- Homology of finite, regular abelian covers
  - Homology of the Milnor fiber of an arrangement
  - Rational homology of smooth, real toric varieties
- Lower central series and Chen Lie algebras
  - The Chen ranks conjecture

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