Purity and flatness for quasicoherent sheaves

by

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What is purity?

Definition
Let \( R \) be a ring and \( 0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0 \) a short exact sequence of right \( R \)-modules. This sequence is called \textit{pure-exact} if one of these two equivalent conditions hold:

1. For every finitely presented right \( R \)-module \( F \), the complex of abelian groups
   \[
   0 \rightarrow \text{Hom}_R(F, A) \rightarrow \text{Hom}_R(F, B) \rightarrow \text{Hom}_R(F, C) \rightarrow 0
   \]
   is exact.

2. For every (finitely presented) left \( R \)-module \( G \), the complex of abelian groups
   \[
   0 \rightarrow A \otimes_R G \rightarrow B \otimes_R G \rightarrow C \otimes_R G \rightarrow 0
   \]
   is exact.
Generalizations

1. In any locally finitely presented abelian (lfp) category $\mathcal{C}$, pure-exactness can be defined via exactness of the functor $\text{Hom}_\mathcal{C}(F, -)$ for every finitely presented object $F$.

2. In any closed monoidal abelian category $\mathcal{D}$, pure-exactness can be defined via exactness of the functor $- \otimes G$ for every (finitely presented) object $G$. 
Let $X$ be a scheme. Then we have the following two Grothendieck categories:

- $\mathcal{O}_X$-$\text{Mod}$, the category of all sheaves of $\mathcal{O}_X$-modules,
- $\text{QCoh}(X)$, the full subcategory of the quasicoherent ones.

$\mathcal{O}_X$-$\text{Mod}$ has a closed (symmetrical) monoidal category structure, which $\text{QCoh}(X)$ inherits.

$\mathcal{O}_X$-$\text{Mod}$ is always lfp. If $X$ is concentrated (= quasicompact & quasiseparated), then $\text{QCoh}(X)$ is lfp.
Introducing purity for sheaves

For both $\mathcal{C} \in \{\mathcal{O}_X\text{-Mod}, \text{QCoC}(X)\}$, we have both ways of introducing purity:

1. using the functors $\text{Hom}_\mathcal{C}(F, -)$ for $F \in \text{fp}(\mathcal{C})$, we will call this purity categorical or $c$-, 
2. using the functors $- \otimes G$ for $G \in \mathcal{C}$, we will call this purity geometric or $g$-, 

and these two notions are not equivalent in general.

Proposition (Enochs-Estrada-Odabaşı)

- Geometrical purity is the same as purity stalk-wise; in $\text{QCoC}(X)$, it also coincides with purity on each open affine set.
- For $X$ concentrated, $c$-purity in $\text{QCoC}(X)$ is stronger than $g$-purity.
The following sequence in $\text{QCoh}(\mathbb{P}^1_k)$ is g-pure-exact, but not c-pure-exact:

$$0 \rightarrow \mathcal{O}(-2) \rightarrow \mathcal{O}(-1)^2 \rightarrow \mathcal{O}(0) \rightarrow 0.$$ 

In detail, passing to an open affine cover:
Let \( X = (\text{Spec } k)^{(\mathbb{N})} \), i.e. countable scheme coproduct of spectra of a field \( k \). As a topological space, \( X \) is a countable discrete set.

\( \mathcal{O}_X\text{-Mod} = \text{Qcoh}(X) = \) countable families of vector spaces over \( k \)-vector spaces, a semisimple category. Therefore both purities are trivial.
When purities coincide

If $X$ is an affine scheme, then $c$-purity $= g$-purity. The converse is

**Proposition (Prest-S.)**

*If $X$ is a concentrated scheme and $c$-purity $= g$-purity in $\text{QCoh}(X)$, then $X$ is affine.*

**Proof.**

Since $X$ is concentrated, the structure sheaf $\mathcal{O}_X$ is finitely presented in $\text{QCoh}(X)$. It is also *flat* (on each open affine), so every short exact sequence ending in it is $g$-pure-exact. If $g$-pure $= c$-pure, then any such sequence splits, so $\mathcal{O}_X$ is projective and $X$ is affine by Serre’s criterion.
Assume further that $X$ is concentrated. Accordingly, we have two types of pure-injective sheaves: c-pure-injective and g-pure-injective (stronger property).

Recall that indecomposable pure-injective objects of a category form a topological space called Ziegler spectrum; for $\text{QCoh}(X)$, g-pure-injectives form a subset of Ziegler spectrum.
Proposition (Prest-S.)

- Every indecomposable $g$-pure-injective $N \in \text{QCoh}(X)$ is the direct image of a unique indecomposable $g$-pure-injective $N' \in \text{QCoh}(U)$, where $U$ is open affine.
- Every indecomposable $g$-pure-injective $N \in \text{QCoh}(X)$ is the coherator of some (non-unique) indecomposable $g$-pure-injective $M \in \mathcal{O}_X\text{-Mod}$. 
- $g$-pure-injectives form a quasicompact closed subset of $Zg(\text{QCoh}(X))$.

A picture appears on the blackboard…
Example: $\text{Zg}(\text{QCoh}(\mathbb{P}^1_k))$

For each closed point $x$ of $X = \mathbb{P}^1_k$, there are the following g-pure-injective indecomposable quasicoherent sheaves:

- completion of $\mathcal{O}_{X,x}$ in its maximal ideal $m_x$,
- “Prüfer” $\mathcal{O}_{X,x}$-module,
- for each $n \in \mathbb{N}$, the finite length sheaf $\mathcal{O}_{X,x}/m_x^n$.

Additionally, there is the g-pure-injective constant sheaf having $k(x)$ everywhere.

Finally, for each $n \in \mathbb{Z}$, the $n$-th twist of the structure sheaf $\mathcal{O}(n)$ is c-pure-injective (but not g-pure-injective).

Note: Because of the line bundles, the Ziegler spectrum is not quasicompact.
Recall that closed subsets of Ziegler spectrum correspond to definable subcategories; let $\mathcal{D}_X$ be the definable subcategory of $\text{QCoh}(X)$ corresponding to indecomposable g-pure-injectives.

This is the subcategory “where purities coincide”:

**Proposition (Prest-S.)**

- A c-pure-injective quasicoherent sheaf is g-pure-injective if and only if it belongs to $\mathcal{D}_X$.
- Any g-pure-exact sequence starting in an object of $\mathcal{D}_X$ is c-pure-exact.
Example

If $X = \mathbb{P}^1_k$, then

$$\mathcal{D}_X = \left\{ m \in \text{QCoh}(X) \mid \forall n \in \mathbb{Z} : \text{Ext}^1_{\text{QCoh}(X)}(\mathcal{O}(n), m) = 0 \right\}$$

$$= \left\{ m \in \text{QCoh}(X) \mid \forall n \in \mathbb{Z} : \text{Hom}^1_{\text{QCoh}(X)}(m, \mathcal{O}(n)) = 0 \right\}.$$ 

Therefore, in this case, $\mathcal{D}_X$ is a torsion class and a right class of a cotorsion pair.
Flat generators
There are (at least) three candidates for what \textit{flat} could mean for \( m \in \text{Q Coh}(X) \):

1. every s.e.s. ending in \( m \) is c-pure-exact; these often do not exist [Estrada-Saorín ’12]

2. every s.e.s. ending in \( m \) is g-pure-exact;

3. tensoring by \( m \) is exact; equivalently, if \( m(U) \) is flat for each open affine \( U \subseteq X \).

Clearly (1) \( \Rightarrow \) (2) and (3) \( \Rightarrow \) (2). If \( X \) is semiseparated, then also (2) \( \Rightarrow \) (3).
We adopt the definition (3): flat = flat on open affines.

If $X$ is quasicompact and semiseparated, then $\text{QCoh}(X)$ has a flat generator [Murfet, Neeman, Positselski...].

A bit surprisingly, this is an equivalence:

**Theorem (S.-Šťovíček)**

*A concentrated scheme $X$ is semiseparated if and only if $\text{QCoh}(X)$ has a flat generator.*
Elementary dual

A useful tool for dealing with purity in modules is the elementary dual $\text{Hom}_R(−, E)$, where $E$ is the injective cogenerator of $\text{Mod}-R$ ($R$ commutative).

For dealing with geometric purity, there is an analogue $\text{Hom}^{qc}(−, ℰ)$, where $ℰ$ is the injective cogenerator of $\text{QCoh}(X)$.

Lemma

Let $C$ be a symmetric closed monoidal Grothendieck category with internal hom functor $\text{HOM}$. If $C$ has a flat generator and $E \in C$ is injective, then the functor $\text{HOM}(−, E)$ is exact.

Therefore for $X$ semiseparated, the elementary dual is exact.
In fact, we again have a sort of converse:

Theorem (S.-Šťovíček)

A concentrated scheme $X$ is semiseparated if and only if for every injective $\mathcal{E} \in \text{QCoh}(X)$, the functor $\mathcal{H}om^{\text{qc}}(-, \mathcal{E})$ is exact.

On the other hand, this does not spoil any nice properties of elementary dual even in the non-semiseparated case.