Valuations and applications

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Contents

p-adic valuation (K. Hensel, 1897, J. Kurschak 1913)

Let p be a fixed prime number. The *p-adic valuation* of an integer n is defined to be

$$\nu_p(n) = \begin{cases} \max\{k \ : \ p^k \mid n\} & n \neq 0 \\ \infty & n = 0. \end{cases}$$

For example, $\nu_3(90) = \nu(2 \cdot 3^2 \cdot 5) = 2$. It is easy to see that ν_p is a function from $\mathbb{Z} \to \mathbb{Z}_{\geq 0} \cup \{\infty\}$ satisfying

(•1) $\nu_p : \mathbb{Z} \to \mathbb{Z} \cup \{\infty\}$.

(•2) $\nu_p(m \cdot n) = \nu_p(m) + \nu_p(n)$.

(•3) $\nu_p(m + n) \geq \min\{\nu_p(m), \nu_p(n)\}$. Moreover, if $\nu_p(m) \neq \nu_p(n)$, then $\nu_p(m + n) = \min\{\nu_p(m), \nu_p(n)\}$.

We can extend the p-adic valuation ν_p to $\mathbb{Q}$ by setting $\nu_p(m/n) := \nu_p(m) - \nu_p(n)$. Then ν_p satisfies (•2) and (•3) as a function from $\mathbb{Q} \to \mathbb{Z} \cup \{\infty\}$.

The *p-adic absolute value* of a rational number r is defined to be $|r|_p := p^{-\nu_p(r)}$ which is used to define a metric on $\mathbb{Q}$. The completion of $\mathbb{Q}$ with respect to this metric leads to the p-adic numbers $\mathbb{Q}_p$.

p-adic numbers serve as a fundamental tool in number theory.

Valuations in commutative algebra (W. Krull 1932, A. Ostrowski 1934)

The idea of the p-adic valuation can be extended to a commutative algebra (1st step was to a PID or a Dedekind domain, then to a general commutative domain).

Definition

Let A be a commutative ring (or a field). A map

$$\nu : A \to \mathbb{Z} \cup \{\infty\}$$

is called a *discrete valuation* if the following holds, for $a, b \in A$,

(v1) $\nu(a) = \infty$ if and only if $a = 0$.

(v2) $\nu(ab) = \nu(a) + \nu(b)$.

(v3) $\nu(a+b) \geq \min\{\nu(a), \nu(b)\}$. Moreover, if $\nu(a) \neq \nu(b)$, then $\nu(a+b) = \min\{\nu(a), \nu(b)\}$.

(v4) Sometimes we assume that ν is surjective. if not surjective, ν could be trivial

Valuations in other subjects

Remarks:

• There are *valuative criteria* for separatedness and properness using DVR in algebraic geometry.

• Valuations are used extensively in tropical algebraic geometry,

• complex analysis, differential geometry, K-theory, and so on.

• Recently, in noncommutative algebra, M. Yakimov uses valuations to understand the automorphism group of some quantized algebras.

Valuations in noncommutative algebra/noncommutative algebraic geometry

H. Hasse (1931) and O.F.G. Schilling (1945) introduced the concept of a valuation in the noncommutative setting (or a noncommutative valuation).

Definition

Let A be a noncommutative ring (or a skew field). A map

$$\nu : A \to \mathbb{Z} \cup \{\infty\}$$

is called a *discrete valuation* or simply a *valuation* if the following hold, for $a, b \in A$,

(nv1) $\nu(a) = \infty$ if and only if $a = 0$.

(nv2) $\nu(ab) = \nu(a) + \nu(b)$.

(nv3) $\nu(a + b) \geq \min\{\nu(a), \nu(b)\}$. Moreover, if $\nu(a) \neq \nu(b)$, then $\nu(a + b) = \min\{\nu(a), \nu(b)\}$.

(nv4) Sometimes we assume that ν is surjective.

Artin's conjecture

Let $\mathbb{k}$ be a base field. In a large part of this talk we assume that $\mathbb{k} = \mathbb{C}$.

In 1996 M. Artin conjectured that the following is a complete list of division $\mathbb{k}$-algebras of transcendence degree 2:

1. PI division algebras.
2. q-rational division algebras

(2a) $Q(\mathbb{k}_q[x, y]) = \mathbb{k}(x, y \ : \ yx = qxy)$ where $q \notin \sqrt[n]{1}$ (q-skew field),

(2b) $Q(A_1) = \mathbb{k}(x, y \ : \ yx - xy = 1)$ (Weyl field),

(2c) the Sklyanin division algebras.

3. q-ruled division algebras

(3a) $\mathbb{k}(E)(t; \sigma)$ where E is an elliptic curve and $|\sigma| = \infty$,

(3b) Q of differential opearators on a curve of higher genus.

Artin's computation on valuations (M. Artin, T. Stafford, M. Van den Bergh 1996)

How to distinguish skew fields? See the number of valuations.

Name	Division rings of tr.deg 2	types	#{discrete valuations }/∼
Weyl field	$Q(A_1(\Bbbk))$	q-rational *	∞ uncountable
q-skew field	$\Bbbk_q(x, y)$, $q \notin \sqrt[n]{1}$	q-rational	∞ countable
Graded version of elliptic type	$\Bbbk(E)(t; \sigma)$ $\lvert\sigma\rvert = \infty$	q-ruled	2
Sklyanin skew field	$Q_{gr}(\mathrm{Skly}_{a,b,c})_0$ generic $[a : b : c]$	q-rational	1
PI	f.g over center	birationally PI	not computed

Table: skew fields of tr.deg 2

Poisson algebras are natural

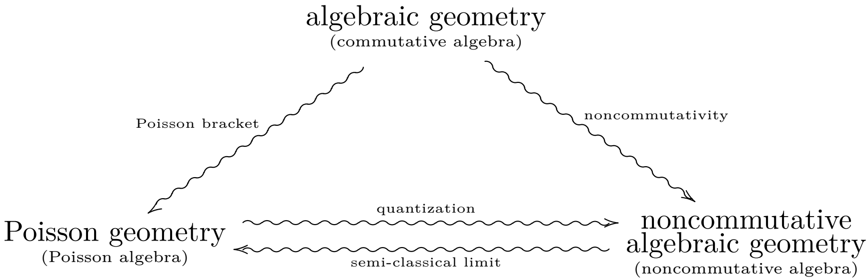
The notion of Poisson bracket was introduced by S.D. Poisson (1809) in the search for integrals of motion in Hamiltonian mechanics.

A *Poisson algebra* P is a commutative algebra together with a Lie bracket $\{-,-\}$ that satisfies the Leibniz rule, that is, the bracket is a derivation of the commutative algebra P on each variable.

Remarks:

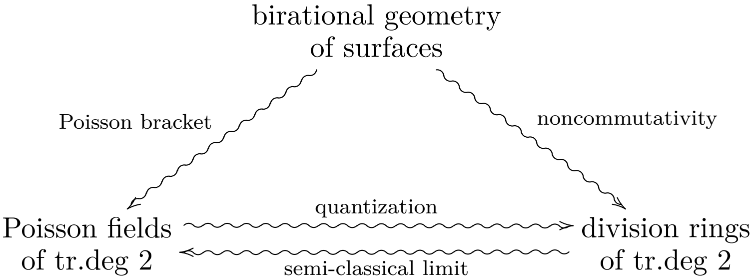
0. Poisson algebra "=" commutative algebra + Lie structure.
1. In operadic language, Pois operad= Com operad $\circ$ Lie operad.
2. A Poisson algebra can be considered as an "infinitesimally noncommutative" algebra.

The relations among algebraic geometry, noncommutative algebraic geometry, and Poisson geometry can be summarized in the following triangle.



Extending Artin's conjecture

We should try to understand both Poisson fields and skew fields of tr.degree 2.



Examples of Poisson algebras

Ex1: Let P be $\Bbbk[x, y]$ with Poisson bracket determined by $\{x, y\} = hxy$ for some $h \in \Bbbk^\times$. This is called the *skew Poisson polynomial algebra*. The corresponding noncommutative algebra is the q-skew polynomial ring $\Bbbk\langle x, y\rangle/(yx - qxy)$.

Ex2: Let P be $\Bbbk[x, y]$ with Poisson bracket determined by $\{x, y\} = 1$. This is called the *Weyl Poisson polynomial algebra*. The Poisson bracket is given by

$$\{f, g\} = \frac{\partial f}{\partial x}\frac{\partial g}{\partial y} - \frac{\partial g}{\partial x}\frac{\partial f}{\partial y} \quad (=: Jac(f, g)).$$

The corresponding noncommutative algebra is the Weyl algebra $\Bbbk\langle x, y\rangle/(yx - xy - 1)$.

Ex3: Let P be $\Bbbk[x, y, z]$ with Poisson bracket determined by

$$\{x, y\} = z^2 + \lambda xy, \{y, z\} = x^2 + \lambda yz, \{z, x\} = y^2 + \lambda zx$$

with $\lambda^3 \neq -1$. This is called the elliptic (or Sklyanin) Poisson polynomial algebra. The corresponding noncommutative algebra is the 3-dimensional Sklyanin algebra $Skly_{a,b,c} := \Bbbk\langle x, y, z\rangle/(R)$ where R is generated by

$$axy + byx + cz^2 = ayz + bzy + cx^2 = azx + bxz + cy^2 = 0$$

where (a, b, c) satisfies some specified condition.

To study noncommutative algebras, one might first understand the corresponding Poisson algebras.

Here is a generalization of Ex3:

Ex4: Let Ω be an element in $\Bbbk[x, y, z]$.
Let P be $\Bbbk[x, y, z]$ with Poisson bracket determined by

$$\{x, y\} = \frac{\partial \Omega}{\partial z}, \{y, z\} = \frac{\partial \Omega}{\partial x}, \{z, x\} = \frac{\partial \Omega}{\partial y}.$$

This Poisson algebra is denoted by $\Bbbk[x, y, z]_{\Omega}$.
In Ex3, we take $\Omega = \frac{1}{3}(x^3 + y^3 + z^3) + \lambda xyz$.

These Poisson algebras are also "unimodular", namely, the modular derivation is zero.

Example 5

The following example is important for this talk.

EX5: Let $\Bbbk[x, y, z]_\Omega$ be defined in Ex4. Then Ω is a Poisson center element. Let $P_{\Omega-\xi}$ be the Poisson factor ring $\Bbbk[x, y, z]_\Omega/(\Omega - \xi)$ where $\xi \in \Bbbk$.

Note that if $\Omega - \xi$ is irreducible, $P_{\Omega-\xi}$ is a Poisson domain of Krull dimension two. Let $Q(P_{\Omega-\xi})$ denote the Poisson field of fractions of $P_{\Omega-\xi}$. (So $Q(P_{\Omega-\xi})$ has tr.degree 2.)

If $\xi = 0$, $P_{\Omega-0}$ is denoted by P_Ω.

Valuations of Poisson algebras

Definition

Let P be a Poisson algebra (or a Poisson field) over $\Bbbk$ and w be any (fixed) integer. A map

$$\nu : P \to \mathbb{Z} \cup \{\infty\}$$

is called a *w-valuation* on P if, for all elements $a, b_1, b_2 \in P$,

(pv1) $\nu(a) = \infty$ if and only if $a = 0$; and $\nu(a) = 0$ for all $a \in \Bbbk^\times := \Bbbk \setminus \{0\}$,

(pv2) $\nu(b_1 b_2) = \nu(b_1) + \nu(b_2)$,

(pv3) $\nu(b_1 + b_2) \geq \min\{\nu(b_1), \nu(b_2)\}$, with equality if $\nu(b_1) \neq \nu(b_2)$,

(new) $\nu(\{b_1, b_2\}) \geq \nu(b_1) + \nu(b_2) - w$.

Let $\mathcal{V}_w(P)$ be the set of w-valuations. Then there is an action

$$\mathcal{V}_w(P) \times \mathrm{Aut}_{Pois\ Alg}(P) \to \mathcal{V}_w(P).$$

Some computations (Huang-Tang-Wang, 2023)

Let $Q(P)$ be a Poisson fraction field of a Poisson domain P. Here is a Poisson analogue of Artin's computation.

(1) If P is the Weyl Poisson polynomial ring (Ex1), then $Q(P)$ has uncountably many 0-valuations.

(2) If P is the skew Poisson polynomial ring (Ex2), then $Q(P)$ has infinite, but countably many, 0-valuations.

(3) If P is Sklyanin Poisson polynomial ring in (Ex3), then $Q(P/\Omega - 1))$ has one 0-valuations.

(4) If $P := \Bbbk[x, y, z]_{\Omega}/(\Omega)$ is a graded version of Sklyanin type (see (Ex5)) , then $Q(P)$ has two 0-valuations.

(5) If $P := \Bbbk[x, y, z]_{\Omega}/(\Omega - 1)$ is as in Ex5 with Ω homogeneous of degree ≥ 5 and having isolated singularity, then $Q(P)$ has no 0-valuation.

Comparison

The first 4 in the first column are function fields of PPS.

Poisson fields of tr.deg 2	Division rings of tr.deg 2	types	#{discrete (0-)valuations}/~
Weyl field	$Q(A_1(\Bbbk))$	rational	∞ uncountable
Skew field	$\Bbbk_q(x,y)$, $q \notin \sqrt[n]{1}$	rational	∞ countable
Graded version of elliptic type	$\Bbbk(E)(t,\sigma)$ $\lvert\sigma\rvert = \infty$	ruled	2
Sklyanin Poisson	$Q_{\text{gr}}(\text{Skly}_{a,b,c})_0$ generic $[a:b:c]$	rational	1
$\deg\Omega \geq 5$ $Q(P_{\Omega-1})$ in Ex5	tr.deg > 2??	high genus ??	0

Table: Comparison between Poisson fields and skew fields of tr.deg 2

Applications of Valuations

Poisson valuations are useful in many topics.

(1) Automorphism problem.
(2) Isomorphism problem.
(3) Embedding problem.
(4) Rigidity of grading.
(5) Rigidity of filtration.
(6) Dixmier problem.
(7) Classification project.
We will only breifly mention (1), (2), and (6) in this talk.

1. Automorphism problem

Automorphism Problem is to compute the automorphism group of a mathematical object. Here we will compute the automorphism group of some Poisson fields.

For the rest of this talk let Ω be a homogeneous element in $\mathbb{k}[x, y, z]$ of degree at least 5. We assume that Ω has isolated singularity. Let $\mathbb{k}[x, y, z]_\Omega$ be the Poisson polynomial algebra given in Ex4. Let $\mathbb{k}(x, y, z)_\Omega$ be the Poisson field of fractions of the Poisson polynomial algebra $\mathbb{k}[x, y, z]_\Omega$. (It has tr.degree 3).

Question

What is the automorphism group of $\mathbb{k}[x, y, z]_\Omega$ (and $\mathbb{k}(x, y, z)_\Omega$)?

Controlling theorem

One key step is the following theorem.
Let P be a Poisson field. The Γ_1-cap of P is defined to be

$$\Gamma_1(P) = \{a \in P \,:\, \nu(a) \geq 0 \;\forall \nu \in \mathcal{V}_1(P)\}.$$

Theorem (Huang-Tang-Wang, 2023)

Let A be a noetherian normal Poisson subalgebra of P such that $Q(A) = P$. Then $\Gamma_1(P) \subseteq A$.

Corollary (Huang-Tang-Wang, 2023)

Let $P = \Bbbk(x, y, z)_\Omega$ and $B = \Bbbk[x, y, z]_\Omega$. Then $B = \Gamma_1(P)$. As a consequence, if A is a noetherian normal Poisson subalgebra of P such that $Q(A) = P$, then $B \subseteq A$.

Here is a main result.

Theorem (Huang-Tang-Wang, 2023)

(1) $\mathrm{Aut}(\Bbbk(x,y,z)_\Omega) = \mathrm{Aut}(\Bbbk[x,y,z]_\Omega)$.

(2) $\mathrm{Aut}(\Bbbk[x,y,z]_\Omega)$ *is a finite subgroup of* $GL_3(\Bbbk)$ *of order bounded above by* $42d(d-3)^2$ *where* $d = \deg\Omega$.

Corollary (Huang-Tang-Wang, 2023)

If $\Omega = x^d + y^d + z^d$ *where* $d \geq 5$, *then there is an exact sequence of groups*

$$1 \to C_{d-3} \times C_{d(d-3)} \to \mathrm{Aut}(\Bbbk[x,y,z]_\Omega) \to S_3 \to 1.$$

Idea of the proof: there is an action $\mathcal{V}_w(P) \times \mathrm{Aut}_{Pois\ Alg}(P) \to \mathcal{V}_w(P)$.

Dixmier property

Definition

Let A be an algebra. We say A has *Dixmier property* if every injective endomorphism of A is bijective.

Example

(1) Every finite dimensional algebra has the Dixmier property.
(2) The Weyl (Poisson) field and the skew (Poisson) field do not have the Dixmier property.

Dixmier Conjecture (1968) states that the nth Weyl algebra has the Dixmier property. (This is stably equivalent to JC by Tsuchimoto 2005, Belov-Kanel and Kontsevich 2007.)
Poisson conjecture (K. Adjamagbo and A. van den Essen 2006): nth Poisson Weyl algebra has the Dixmier property.
Then DC $\Leftrightarrow$ JC $\Leftrightarrow$ PC.

Question

Is there a criterion for the Dixmier property?

Here is a result about the Dixmier property.

Theorem (Huang-Tang-Wang, 2023)

Retain the above notation. Suppose Ω has isolated singularity and of degree at least 5.

(1) $\mathbb{k}[x, y, z]_\Omega$ *has the Dixmier property.*

(2) $\mathbb{k}(x, y, z)_\Omega$ *has the Dixmier property.*

2. Isomorphism problem

Isomorphism problem is to determine whether or not two mathematical objects in the same family are isomorphic (or equivalent).

For example, give two Poisson fields, how can we determine if these are isomorphic or not?

Definition

If $b \in \mathbb{k}(x, y)$, we let $K\{b\}$ denote the (rational) Poisson field $\mathbb{k}(x, y : \{x, y\} = b)$. (tr.degree 2)

Question

Determine when $K\{b\}$ and $K\{b'\}$ are isomorphic for two nonzero elements b, b' in $\mathbb{k}(x, y)$.

Example

(1) $K\{f(x)\} \cong K\{1\}$ where the latter is the Weyl Poisson field.
(2) $K\{x^{a+1}y^{b+1}\} \cong K\{x^g xy\}$ where $g = \gcd(a, b)$.

Recall that, if $f(x) \in \Bbbk[x] \setminus \Bbbk$, we let $K\{xyf(x)\}$ be the Poisson field $\Bbbk(x, y \;:\; \{x, y\} = xyf(x))$.
Using valuations we can show that

Theorem (Goodearl, 2024)

Let $g(x)$ and $h(x)$ be two polynomials of positive degree. Then $K\{xyg(x)\} \cong K\{xyh(x)\}$ if and only there are $a \in \Bbbk^\times$ and $b \in \Bbbk$ such that

$$a^{-1}(ax - b)g(ax - b) = \pm xh(x).$$

Valuation method works for other algebraic systems

Remark: If $\mathbb{O}$ is an operad such that there is an operadic morphism $f\colon Com \to \mathbb{O}$, then valuation method works for $\mathbb{O}$-algebras.

There are many classes of operads that have such a morphism.

Some of which are called Lie-admissible operads.

For example, valuation method can be applied to n-Lie Poisson algebras.

Thank you!