n-Abelian categories through functor categories

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Maurice Auslander Distinguished Lectures and International Conference
April 26, 2024

Based on ongoing work supervised by Alex Martsinkovsky

Conventions for this talk

- $\mathcal{C}$ is an additive and idempotent complete category.
- n is a positive integer.

A **(right) $\mathcal{C}$-module** is a contravariant additive functor from $\mathcal{C}$ to Ab.

A $\mathcal{C}$-module F is **finitely presented** if there is an exact sequence

$$\mathcal{C}(-,X) \xrightarrow{\mathcal{C}(-,f)} \mathcal{C}(-,Y) \longrightarrow F \longrightarrow 0$$

for some morphism $X \xrightarrow{f} Y$ in $\mathcal{C}$.

The category of finitely presented $\mathcal{C}$-modules is denoted by **$\operatorname{mod}\mathcal{C}$**.

The projective objects of $\operatorname{mod}\mathcal{C}$ are given by $\mathcal{C}(-,X)$, where $X \in \mathcal{C}$.

$*$ By taking $\mathcal{C}^{\mathrm{op}}$ in place of $\mathcal{C}$, we obtain the category **$\operatorname{mod}\mathcal{C}^{\mathrm{op}}$** of finitely presented **left** $\mathcal{C}$-modules.

We say that $\mathcal{C}$ is **right coherent** when $\operatorname{mod} \mathcal{C}$ is abelian,

$\mathcal{C}$ is **left coherent** when $\operatorname{mod} \mathcal{C}^{\mathrm{op}}$ is abelian.

It was independently proved by Auslander (1965) and Freyd (1965) that

$\operatorname{mod} \mathcal{C}$ is abelian if and only if $\mathcal{C}$ has has weak kernels,

$\operatorname{mod} \mathcal{C}^{\mathrm{op}}$ is abelian if and only if $\mathcal{C}$ has has weak cokernels.

Idea: To describe properties of $\mathcal{C}$ in terms of $\operatorname{mod} \mathcal{C}$ and $\operatorname{mod} \mathcal{C}^{\mathrm{op}}$, and vice versa.

Goal: To describe properties of an n-abelian category $\mathcal{C}$ in terms of $\operatorname{mod}\mathcal{C}$ and $\operatorname{mod}\mathcal{C}^{\text{op}}$.

Advantage: We could understand n-abelian categories through abelian categories.

The category $\mathcal{C}$ is **abelian** if and only if it satisfies the following axioms:

(A1) $\mathcal{C}$ has kernels.

(A1$^{\text{op}}$) $\mathcal{C}$ has cokernels.

(A2) Every monomorphism in $\mathcal{C}$ is the kernel of its cokernel.

(A2$^{\text{op}}$) Every epimorphism in $\mathcal{C}$ is the cokernel of its kernel.

This is the case $\boldsymbol{n = 1}$ of the more general notion of an ***n*-abelian category** defined by Jasso (2016).

An ***n*-kernel** for $X \xrightarrow{f} Y$ is a sequence

$$X_n \xrightarrow{f_n} \cdots \xrightarrow{f_2} X_1 \xrightarrow{f_1} X$$

of morphisms in $\mathcal{C}$ with the property that

$$0 \longrightarrow \mathcal{C}(-, X_n) \xrightarrow{\mathcal{C}(-,f_n)} \cdots \xrightarrow{\mathcal{C}(-,f_2)} \mathcal{C}(-, X_1) \xrightarrow{\mathcal{C}(-,f_1)} \mathcal{C}(-, X) \xrightarrow{\mathcal{C}(-,f)} \mathcal{C}(-, Y)$$

is exact.

An ***n*-cokernel** for $X \xrightarrow{f} Y$ is a sequence

$$Y \xrightarrow{g_1} Y_1 \xrightarrow{g_2} \cdots \xrightarrow{g_n} Y_n$$

of morphisms in $\mathcal{C}$ with the property that

$$0 \longrightarrow \mathcal{C}(Y_n, -) \xrightarrow{\mathcal{C}(g_n, -)} \cdots \xrightarrow{\mathcal{C}(g_2, -)} \mathcal{C}(Y_1, -) \xrightarrow{\mathcal{C}(g_1, -)} \mathcal{C}(Y, -) \xrightarrow{\mathcal{C}(f, -)} \mathcal{C}(X, -)$$

is exact.

The category $\mathcal{C}$ is ***n*-abelian** if and only if it satisfies the following axioms:

(A1) $\mathcal{C}$ has n-kernels.

(A1$^{\mathrm{op}}$) $\mathcal{C}$ has n-cokernels.

(A2) For every monomorphism $X \xrightarrow{f} Y$ in $\mathcal{C}$ and for every n-cokernel

$$Y \xrightarrow{g_1} Y_1 \xrightarrow{g_2} \cdots \xrightarrow{g_n} Y_n$$

of f, the sequence

$$X \xrightarrow{f} Y \xrightarrow{g_1} \cdots \xrightarrow{g_{n-1}} Y_{n-1}$$

is an n-kernel of g_n.

(A2$^{\mathrm{op}}$) For every epimorphism $X \xrightarrow{f} Y$ in $\mathcal{C}$ and for every n-kernel

$$X_n \xrightarrow{f_n} \cdots \xrightarrow{f_2} X_1 \xrightarrow{f_1} X$$

of f, the sequence

$$X_{n-1} \xrightarrow{f_{n-1}} \cdots \xrightarrow{f_1} X \xrightarrow{f} Y$$

is an n-cokernel of f_n.

Axioms of an n-abelian category $\mathcal{C}$	In terms of $\operatorname{mod}\mathcal{C}$ and $\operatorname{mod}\mathcal{C}^{\mathrm{op}}$
(A1)	(F1)
(A1$^{\mathrm{op}}$)	(F1$^{\mathrm{op}}$)
(A2)	(F2)
(A2$^{\mathrm{op}}$)	(F2$^{\mathrm{op}}$)

* The following axioms are equivalent:

(A1) $\mathcal{C}$ has n-kernels.

(F1) $\mathcal{C}$ is right coherent and $\operatorname{gl.dim}(\operatorname{mod}\mathcal{C}) \leqslant n+1$.

* Dually, the following are equivalent:

(A1$^{\text{op}}$) $\mathcal{C}$ has n-cokernels.

(F1$^{\text{op}}$) $\mathcal{C}$ is left coherent and $\operatorname{gl.dim}(\operatorname{mod}\mathcal{C}^{\text{op}}) \leqslant n+1$.

* If $\mathcal{C}$ satisfies (A1) and (A1$^{\mathrm{op}}$), then the following axioms are equivalent:

(A2) For every monomorphism (...).

(F2) Every $F \in \operatorname{mod} \mathcal{C}$ with $\operatorname{pd} F \leqslant 1$ is n-torsion free.

* Dually, the following are equivalent:

(A2$^{\mathrm{op}}$) For every epimorphism (...).

(F2$^{\mathrm{op}}$) Every $F \in \operatorname{mod} \mathcal{C}^{\mathrm{op}}$ with $\operatorname{pd} F \leqslant 1$ is n-torsion free.

$F \in \operatorname{mod} \mathcal{C}$ is ***n*-torsion free** if $\operatorname{Ext}^i(\operatorname{Tr} F, \mathcal{C}(X, -)) = 0$ for all $X \in \mathcal{C}$ and $1 \leqslant i \leqslant n$.

Theorem (G.)

An additive and idempotent complete category $\mathcal{C}$ is n-abelian if and only if $\mathcal{C}$ satisfies the following axioms:

(F1) $\mathcal{C}$ is right coherent and $\operatorname{gl.dim}(\operatorname{mod}\mathcal{C}) \leqslant n+1$.

(F1$^{\mathrm{op}}$) $\mathcal{C}$ is left coherent and $\operatorname{gl.dim}(\operatorname{mod}\mathcal{C}^{\mathrm{op}}) \leqslant n+1$.

(F2) Every $F \in \operatorname{mod}\mathcal{C}$ with $\operatorname{pd} F \leqslant 1$ is n-torsion free.

(F2$^{\mathrm{op}}$) Every $F \in \operatorname{mod}\mathcal{C}^{\mathrm{op}}$ with $\operatorname{pd} F \leqslant 1$ is n-torsion free.

* Either $\operatorname{gl.dim}(\operatorname{mod}\mathcal{C}) = 0$ or $n+1$.

Fact. When $n = 1$, we can replace the axioms (A2) and ($A2^{op}$) by:

($A2_*$) Every monomorphism is a kernel.

($A2_*^{op}$) Every epimorphism is a cokernel.

Question. What about for an arbitrary n?

* If $\mathcal{C}$ is right and left coherent, then the following axioms are equivalent:

(F2) Every $F \in \operatorname{mod} \mathcal{C}$ with $\operatorname{pd} F \leqslant 1$ is n-torsion free.

($F2_*$) Every m-spherical object in $\operatorname{mod} \mathcal{C}$ is a syzygy, for all $1 \leqslant m \leqslant n$.

* Dually, the following are equivalent:

($F2^{op}$) Every $F \in \operatorname{mod} \mathcal{C}^{op}$ with $\operatorname{pd} F \leqslant 1$ is n-torsion free.

($F2_*^{op}$) Every m-spherical object in $\operatorname{mod} \mathcal{C}^{op}$ is a syzygy, for all $1 \leqslant m \leqslant n$.

$F \in \operatorname{mod} \mathcal{C}$ is ***m*-spherical** if $\operatorname{pd} F \leqslant m$ and $\operatorname{Ext}^i(F, \mathcal{C}(-, X)) = 0$ for all $X \in \mathcal{C}$ and $1 \leqslant i \leqslant m-1$.

Let m be a positive integer. An ***m*-segment** in $\mathcal{C}$ is a sequence

$$X_m \xrightarrow{f_m} \cdots \xrightarrow{f_2} X_1 \xrightarrow{f_1} X_0$$

of morphisms in $\mathcal{C}$ for which

$$0 \longrightarrow \mathcal{C}(-,X_m) \xrightarrow{\mathcal{C}(-,f_m)} \cdots \xrightarrow{\mathcal{C}(-,f_2)} \mathcal{C}(-,X_1) \xrightarrow{\mathcal{C}(-,f_1)} \mathcal{C}(-,X_0)$$

and

$$\mathcal{C}(X_0,-) \xrightarrow{\mathcal{C}(f_1,-)} \mathcal{C}(X_1,-) \xrightarrow{\mathcal{C}(f_2,-)} \cdots \xrightarrow{\mathcal{C}(f_m,-)} \mathcal{C}(X_m,-)$$

are exact.

* A 1-segment is the same as a monomorphism.

Let m be a positive integer. An ***m*-cosegment** in $\mathcal{C}$ is a sequence

$$Y_0 \xrightarrow{g_1} Y_1 \xrightarrow{g_2} \cdots \xrightarrow{g_m} Y_m$$

of morphisms in $\mathcal{C}$ such that

$$0 \longrightarrow \mathcal{C}(Y_m, -) \xrightarrow{\mathcal{C}(g_m, -)} \cdots \xrightarrow{\mathcal{C}(g_2, -)} \mathcal{C}(Y_1, -) \xrightarrow{\mathcal{C}(g_1, -)} \mathcal{C}(Y_0, -)$$

and

$$\mathcal{C}(-, Y_0) \xrightarrow{\mathcal{C}(-, g_1)} \mathcal{C}(-, Y_1) \xrightarrow{\mathcal{C}(-, g_2)} \cdots \xrightarrow{\mathcal{C}(-, g_m)} \mathcal{C}(-, Y_m)$$

are exact.

∗ A 1-cosegment is the same as an epimorphism.

* If $\mathcal{C}$ is right and left coherent, then the following axioms are equivalent:

($F2_*$) Every m-spherical object in $\operatorname{mod}\mathcal{C}$ is a syzygy, for all $1 \leqslant m \leqslant n$.
($A2_*$) Every m-segment in $\mathcal{C}$ is an m-kernel, for all $1 \leqslant m \leqslant n$.

* Dually, the following are equivalent:

($F2_*^{op}$) Every m-spherical object in $\operatorname{mod}\mathcal{C}^{op}$ is a syzygy, for all $1 \leqslant m \leqslant n$.
($A2_*^{op}$) Every m-cosegment in $\mathcal{C}$ is an m-cokernel, for all $1 \leqslant m \leqslant n$.

Theorem (G.)

An additive and idempotent complete category $\mathcal{C}$ is n-abelian if and only if $\mathcal{C}$ satisfies the following axioms:

(A1) $\mathcal{C}$ has n-kernels.

(A1$^{\mathrm{op}}$) $\mathcal{C}$ has n-cokernels.

(A2$_*$) Every m-segment in $\mathcal{C}$ is an m-kernel, for all $1 \leqslant m \leqslant n$.

(A2$_*^{\mathrm{op}}$) Every m-cosegment in $\mathcal{C}$ is an m-cokernel, for all $1 \leqslant m \leqslant n$.

Axioms of an n-abelian category $\mathcal{C}$	In terms of $\operatorname{mod}\mathcal{C}$ and $\operatorname{mod}\mathcal{C}^{\mathrm{op}}$	Case $\mathcal{C} = \operatorname{proj}\Lambda$, where Λ is a ring
(A1)	(F1)	(R1)
(A1$^{\mathrm{op}}$)	(F1$^{\mathrm{op}}$)	(R1$^{\mathrm{op}}$)
(A2)	(F2)	(R2)
(A2$^{\mathrm{op}}$)	(F2$^{\mathrm{op}}$)	(R2$^{\mathrm{op}}$)

Theorem (G.)

Let Λ be a ring. The category $\operatorname{proj}\Lambda$ of finitely generated projective Λ-modules is n-abelian if and only if Λ satisfies the following axioms:

(R1) Λ is right coherent and $\operatorname{gl.dim}(\operatorname{mod}\Lambda) \leqslant n+1$.

(R1$^{\mathrm{op}}$) Λ is left coherent and $\operatorname{gl.dim}(\operatorname{mod}\Lambda^{\mathrm{op}}) \leqslant n+1$.

(R2) Every $M \in \operatorname{mod}\Lambda$ with $\operatorname{pd} M \leqslant 1$ is n-torsion free.

(R2$^{\mathrm{op}}$) Every $M \in \operatorname{mod}\Lambda^{\mathrm{op}}$ with $\operatorname{pd} M \leqslant 1$ is n-torsion free.

Theorem (G.)

There is a bijective correspondence between the equivalence classes of n-abelian categories with additive generators and the Morita equivalence classes of rings satisfying the axioms (R1), (R1$^{\text{op}}$), (R2) and (R2$^{\text{op}}$). The correspondence is given as follows:

$$\left\{ \begin{array}{c} n\text{-Abelian categories} \\ \text{with additive generators} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{Rings satisfying} \\ \text{(R1), (R1}^{\text{op}}\text{), (R2), (R2}^{\text{op}}\text{)} \end{array} \right\}$$

$$\mathcal{C} = \operatorname{add} X \longmapsto \operatorname{End}(X)$$

$$\operatorname{proj} \Lambda \longleftarrow\!\shortmid \Lambda$$

The higher Auslander correspondence is a special case.

$$\left\{\begin{array}{c}\text{n-Abelian categories}\\ \text{with additive generators}\\ \left\{\begin{array}{c}\text{n-Cluster tilting subcategories}\\ \text{with additive generators of}\\ \text{categories of finitely presented}\\ \text{modules over Artin algebras}\end{array}\right\}\end{array}\right\} \begin{array}{c}\longleftrightarrow\\ \longleftrightarrow\end{array} \left\{\begin{array}{c}\text{Rings satisfying}\\ \text{(R1), (R1}^{\text{op}}\text{), (R2), (R2}^{\text{op}}\text{)}\\ \left\{\text{n-Auslander algebras}\right\}\end{array}\right\}$$

Thank you!