Representations of 4-dimensional Sklyanin algebras through Poisson geometry

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Maurice Auslander International Conference
October 26, 2022
Contents

1 Introduction

2 Noncommutative Projective Algebraic Geometry

3 Representations of Polynomial Identity (PI) algebras

4 Poisson Geometry

5 Conclusions
I. Introduction
Background

- 4-dimensional Sklyanin algebras were introduced in the early 1980s by Sklyanin as a family of graded algebras in terms of Baxter’s $2 \times 2$ solution to the Yang-Baxter equation.

- Artin and Schelter in the late 1980s classified regular graded algebras of global dimension 3, among which the Type A algebras are now called the 3-dimensional Sklyanin algebras.

- These algebras were the most difficult class to study as they are quite tough to analyze with traditional Gröbner basis techniques. So, projective geometric data was assigned to the algebras, in order to analyze their ring-theoretic and homological behaviors.
I. Noncommutative Projective Algebraic Geometry
Setup I

- 4-dim’l Sklyanin algebras $S := S(\alpha, \beta, \gamma) = \mathbb{C}\langle x_0, x_1, x_2, x_3 \rangle$ subject to

\[
\begin{align*}
  x_0x_1 - x_1x_0 &= \alpha (x_2x_3 + x_3x_2), \\
  x_0x_1 + x_1x_0 &= x_2x_3 - x_3x_2, \\
  x_0x_2 - x_2x_0 &= \beta (x_3x_1 + x_1x_3), \\
  x_0x_2 + x_2x_0 &= x_3x_1 - x_1x_3, \\
  x_0x_3 - x_3x_0 &= \gamma (x_1x_2 + x_2x_1), \\
  x_0x_3 + x_3x_0 &= x_1x_2 - x_2x_1.
\end{align*}
\]

- $\alpha + \beta + \gamma + \alpha\beta\gamma = 0$ with $\alpha, \beta, \gamma \neq 0, \pm 1$.

- $S = \bigoplus_{i \geq 0} S_i = \mathbb{C} \oplus S_1 \oplus S_2 \oplus \cdots$, $S_1 = \text{span}_\mathbb{C}\{x_0, x_1, x_2, x_3\}$.

- Canonical regular central elements

\[
g_1 = -x_0^2 + x_1^2 + x_2^2 + x_3^3, \quad g_2 = x_1^2 + \frac{1 + \alpha}{1 - \beta} x_2^2 + \frac{1 - \alpha}{1 + \gamma} x_3^2.
\]
Artin-Schelter regular algebras

Definition (Artin-Schelter, 1987)

A connected graded algebra $A = \bigoplus_{i \geq 0} A_i$ with $A_0 = \mathbb{C}$ is Artin-Schelter regular of dimension $d$ if

1. $A$ has finite global dimension $d$,
2. \[ \text{Ext}_A^i (\mathbb{C}, A) = \begin{cases} 0 & \text{if } i \neq d \\ \mathbb{C} & \text{if } i = d, \end{cases} \]
3. $A$ has polynomial growth.

In particular, $S$ is an Artin-Schelter regular algebras of dimension 4.
Geometric data of $S$

$S$ is a geometric ring with geometric data $(\hat{E}, \mathcal{L}, \sigma)$:

- **Point scheme**: a smooth elliptic curve $E := \mathbb{V}(\phi_1, \phi_2) \subset \mathbb{P}^3$, where
  \[
  \phi_1 = x_0^2 + x_1^2 + x_2^2 + x_3^2, \quad \phi_2 = \frac{1 - \gamma}{1 + \alpha} x_1^2 + \frac{1 + \gamma}{1 - \beta} x_2^2 + x_3^2
  \]
  plus 4 additional points $\{e_0, e_1, e_2, e_3\}$.

- **Canonical line bundle**: $\mathcal{L} = \mathcal{O}_E(1)$.

- **Translation automorphism**: $\sigma \in \text{Aut}_\mathbb{C}(E)$ such that
  \[
  \sigma(p) = p + \tau, \quad \text{for any } p \in E
  \]
  for some point $\tau \in E$. 

Twisted homogeneous coordinate rings

**Definition (Artin-Van den Bergh, 1990)**

The twisted homogeneous coordinate ring constructed from the geometric data \((E, \mathcal{L}, \sigma)\) is

\[
B := B(E, \mathcal{L}, \sigma) = \bigoplus_{i \geq 0} H^0(E, \mathcal{L}_i)
\]

where \(\mathcal{L}_0 = \mathcal{O}_E\) and \(\mathcal{L}_i = \mathcal{L} \otimes \mathcal{L}^\sigma \otimes \cdots \mathcal{L}^\sigma_{i-1}\) for \(i \geq 1\). Here \(\mathcal{L}^\sigma\) is the pullback of \(\mathcal{L}\) along \(\sigma : \mathcal{L} \to \mathcal{L}\). The multiplication of \(B\) is given by the composition of maps

\[
H^0(E, \mathcal{L}_i) \otimes H^0(E, \mathcal{L}_j) \xrightarrow{1 \otimes \sigma^i} H^0(E, \mathcal{L}_i) \otimes H^0(E, (\mathcal{L}^\sigma_i)_j) \xrightarrow{\text{mult.}} H^0(E, \mathcal{L}_i \otimes (\mathcal{L}^\sigma_i)_j) \cong H^0(E, \mathcal{L}_{i+j})
\]
The quotient graded category

**Theorem (J.P. Serre, 1955)**

- \( R \): graded commutative algebra finitely generated in degree 1
- \( \text{qgr}(R) := \text{gr}(R)/\text{tors}(R) \): the quotient graded category

Then there is an equivalence of categories

\[
\text{coh}(X) \cong \text{qgr}(R)
\]

**Theorem (Artin-Van den Bergh, 1990)**

- \((E, \mathcal{L}, \sigma)\): geometric data
- \( R = B(E, \mathcal{L}, \text{id}) \): homogeneous coordinate ring
- \( B = B(E, \mathcal{L}, \sigma) \): twisted homogeneous coordinate ring

\[
\text{qgr}(B) \cong \text{qgr}(R) \cong \text{coh}(E)
\]
Properties of $S$

Theorem (Smith-Stafford, 1992 and Smith 1993)

For any 4-dimensional Sklyanin algebra $S(\alpha, \beta, \gamma)$ with geometric data $(\hat{E}, \mathcal{L}, \sigma)$, we have

1. There is a ring surjection $S(\alpha, \beta, \gamma) \twoheadrightarrow B(E, \mathcal{L}, \sigma)$, whose kernel is generated by the two regular central elements $g_1, g_2$.

2. $S$ has nice ring-theoretic and homological properties.

3. $S$ is module-finite over its center if and only if $|\sigma| < \infty$. In this case, the PI degree of $S$ is $|\sigma|$. 
II. Representations of Polynomial Identity (PI) algebras
Setup II

- $Z = Z(S)$, $Y = \text{maxSpec}(Z)$.
- Azumaya locus of $S$: $\mathcal{A}_S = \{m \in Y | S_m \text{ Azumaya over } Z_m\}$.
- $S$ is module-finite over $Z$ with PI-deg($S$) = $|\tau| =: n$.
- $\text{Irr}(S)$: iso-classes of all $S$-simples.
- $\text{Irr}_p(S)$: iso-classes of all $S$-simples of dimension $p$.

\[
\Psi : \text{Irr}(S) \rightarrow Y \quad \quad V \rightarrow \text{Ann}_Z(V).
\]
Theorem (Brown-Goodearl, 1997)

Suppose $S$ is prime noetherian and module-finite over its center $Z$. Assume its center $Z$ is an affine algebra over $\mathbb{C}$.

- The maximal $\mathbb{C}$-dim of $S$-simples = PI-deg of $S$ ($= n$).
- For any $V \in \text{Irr}(S)$, $\dim_{\mathbb{C}}(V) = n$ if and only if $\Psi(V) \in A_S$. In this case, $\Psi : \text{Irr}(S) \rightarrow Y$ yields a one-to-one correspondence between $\text{Irr}_n(S)$ and $A_S$.
- If $\text{gldim}(S) < \infty$, then $A_S \subseteq Y^{\text{smooth}}$. Moreover if $A_S$ has codimension $\leq 2$ in $Y$, then $A_S = Y^{\text{smooth}}$. 

Representation of PI algebras
III. Poisson Geometry
Poisson algebras

Definition (Siméon Poisson, 1800)

- A Poisson algebra $Z$ is a commutative algebra together with a bilinear map

$$\{−, −\} : Z \times Z → Z$$

that is both a Lie bracket and a biderivation.

- Suppose $(Z, \{−, −\})$ is a Poisson algebra. Then $Y = \text{maxSpec}(Z)$ is an affine variety with the bivector

$$\pi = \{−, −\} ∈ \wedge^2(TY)$$

satisfying a vanishing Schouten-Nijenhuis bracket $[\pi, \pi]_S = 0$. 
There are 3 different stratifications on an affine Poisson variety $Y = \text{maxSpec}(Z)$. Take any $m \in Y$.

- **Rank stratification:**
  
  $$\mathcal{R}(m) = \{n \mid \text{rank}(-,-)_n = \text{rank}(-,-)_m\}.$$

- **Symplectic core stratification:**
  
  $$\mathcal{C}(m) = \{n \in Y \mid \mathcal{P}(n) = \mathcal{P}(m)\},$$

  where the symplectic core $\mathcal{P}(m)$ of $m$ is the maximal Poisson ideal inside $m$.

- **Symplectic leaf stratification:** $\mathcal{L}(m)$

  $$Y = \underbrace{Y^{\text{smooth}}} \sqcup \underbrace{Y^{\text{sing}}}$$

  symplectic foliations  closed Poisson submanifold
De Concini-Kac-Procesi pioneered the applications of Poisson geometry in the representation theory of PI algebras, for the cases of quantized universal enveloping algebras and quantum function algebras at roots of unity.

This approach was axiomatized by Brown-Gordon in the theory of Poisson orders and was applied to other families of algebras with PBW bases, such as the symplectic reflection algebras.
Theorem (De Concini-Kac-Procesi, 1992 & Brown-Gordon, 2002)

Suppose $S$ is a Poisson $\mathbb{Z}$-order, namely $S$ is module-finite over $\mathbb{Z}$ and there is a $\mathbb{C}$-linear map $\partial : \mathbb{Z} \to \text{Der}_\mathbb{C}(S/\mathbb{Z})$ such that $\partial|\mathbb{Z}$ equips $\mathbb{Z}$ with a Poisson structure.

If $m, n \in Y = \text{maxSpec}(\mathbb{Z})$ are in the same symplectic core, then there is an isomorphism between the corresponding finite-dimensional $\mathbb{C}$-algebras

$$S/(mS) \cong S/(nS)$$

In this case, for $\Psi : \text{Irr}(S) \to Y$,

$$\Psi^{-1}(m) \cong \Psi^{-1}(n)$$
IV. Conclusions
Summary

- Representation of PI Sklyanin algebras can be viewed in parallel with representations of quantum groups at roots of unity and modular representations of Lie algebras. This case is substantially harder than the non-PI case.
- In this work, we unify the algebro-geometric methods for Sklyanin algebras using the geometry of an elliptic curve with Poisson geometric methods for quantum groups at roots of unity, to
  1. construct nontrivial structures of Poisson orders on all PI 4-dimensional Sklyanin algebras, and
  2. explicitly classify their irreducible representations and describe their dimensions.
Setup III

- $\mathbb{C}$: base field.
- $S$ is module-finite over its center $Z$; $\text{PI-deg}(S) = |\sigma| = n$.
- $Y = \text{maxSpec}(Z)$, $Y^{\text{smooth}}$, $Y^{\text{sing}}$.
- $Y_{\gamma_1,\gamma_2} = Y \cap \mathbb{V}(g_1 - \gamma_1, g_2 - \gamma_2)$, for any $\gamma_1, \gamma_2 \in \mathbb{C}$. 
Theorem (Smith-Tate, 1994)

The center $Z$ of $S$ is given by

$$Z = \mathbb{C}[z_0, z_1, z_2, z_3, g_1, g_2]/(F_1, F_2)$$

where

- $z_0, z_1, z_2, z_3$: 4 algebraically independent elements of deg $n$
- $g_1, g_2$: 2 canonical central regular elements of degree 2
- $F_1, F_2$: 2 homogeneous relations of degree $2n$

namely

$$Y = \mathbb{V}(F_1, F_2) \subset \mathbb{A}^6_{(z_0, z_1, z_2, z_3, g_1, g_2)}$$
Conclusions: Part I

Theorem (Walton-Yakimov-W. 2021)

Suppose $S$ is a 4-dimensional Sklyanin algebra, which is module-finite over its center $Z = k[z_0, z_1, z_2, z_3, g_1, g_2]/(F_1, F_2)$. The following hold:

1. $S$ admits a nontrivial structure of Poisson order.
2. The induced Poisson structure on $Z$ is of Jacobian form in terms of the two potentials taken to be $F_1$ and $F_2$:

$$\{z_i, z_j\} = (-1)^{i+j} \det \left( \frac{\partial (F_1, F_2)}{\partial (z_0, \ldots, \hat{z}_i, \ldots, \hat{z}_j, \ldots, z_3)} \right)$$

3. $g_1, g_2$ lie in the Poisson center of the Poisson algebra $Z$. 
Conclusions: Part II

Theorem (Walton-Yakimov-W. 2021)

1. The corresponding symplectic core stratification of the Poisson variety $Y$ are
   
   - **2-dimensional cores:**
     
     $$\text{(}Y_{\gamma_1,\gamma_2}\text{)}^\text{smooth} := \text{smooth}\ Y_{\gamma_1,\gamma_2} \setminus \text{(}Y_{\gamma_1,\gamma_2}\text{)}^\text{sing} \text{ for } \gamma_1, \gamma_2 \in \mathbb{C}$$

   - **0-dimensional cores:** points on which the Poisson bracket on $Y$ vanishes
     
     $$Y_{0}^\text{sym} := \bigcup_{\gamma_1,\gamma_2\in\mathbb{C}} (Y_{\gamma_1,\gamma_2})^\text{sing}$$

2. $Y^\text{sing} \subseteq Y_{0}^\text{sym}$, with strict containment when $n$ is odd, and they both have codimension $\geq 2$ in $Y$.

3. The Azumaya locus of $S$ coincides with the smooth locus $Y \setminus Y^\text{sing}$ in $Y$. 
Conclusions: Part III

**Theorem (Walton-Yakimov-W. 2021)**

Suppose $\tau$ is of finite order $n > 4$. Then $S$ has irreducible representations of each dimension $1, 2, 3, \ldots, n$ if $n$ is odd, and of each dimension $1, 2, 3, \ldots, \frac{n}{2}, n$ if $n$ is even.

**Remark**

There is a representation-theoretic connection between the 4-dimensional Sklyanin algebra at points of finite order and the quantized enveloping algebra $U_q(SU(2))$ at $q$ a root of unity. Roche and Arnaudon proved that when $q$ is a root of unity of order $n$, then $U_q(SU(2))$ has $d$-dimensional irreducible representations for all $1 \leq d \leq n$ when $n$ is odd, and has $d$-dimensional irreducible representations for all $1 \leq d \leq \frac{n}{2}$ and $d = n$ when $n$ is even.
History of the 4-dimensional Sklyanin algebras

1982-1983, Sklyanin defined a family of graded algebras $S(E, \tau)$ in terms of Baxter’s $2 \times 2$ solution to the Yang-Baxter equation.

1985-1986, Artin and Schelter classified regular graded algebras of global dimension 3, among which the Type A algebras are now called the 3-dimensional Sklyanin algebras.

1989, Smith and Stafford proved that $S(E, \tau)$ has excellent homological and ring-theoretical properties using the algebro-geometric methods introduced by Artin, Tate and Van den Bergh.

1989, Odesskii and Feigin defined higher dimensional Sklyanin algebras, depending on an elliptic curve $E$ and a point $\tau \in E$.

1993, Levasseur and Smith classified point, line and plane modules over $S(E, \tau)$.

1993, Smith and Staniszkis classified all finite-dimensional simple modules over $S(E, \tau)$ with $|\tau| = \infty$.

1993, Smith classified all fat point modules over $S(E, \tau)$ with $|\tau| < \infty$ and proved that $S(E, \tau)$ satisfies a polynomial identity of degree $2|\tau|$.

1994, Smith and Tate described the center of the 3-dimensional and 4-dimensional Sklyanin algebras with $|\tau| < \infty$. 
Thank you very much for your attention!