On irreducible morphisms and Auslander–Reiten triangles in the stable category of modules over a repetitive algebra

Jose A. Velez-Marulanda
Valdosta State University

(joint work with Y. Calderon-Henao and H. Giraldo)

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In this talk:

* $k$ is a field with $k = \overline{k}$;
* $\Lambda$ denotes a finite-dimensional $k$-algebra.

Repetitive algebras (D. Hughes and J. Warchalski, 1993)

The repetitive algebra $\hat{\Lambda}$ of $\Lambda$ is defined as follows:

* Underlying $k$-vector space: $\hat{\Lambda} = \bigoplus_{i \in \mathbb{Z}} \Lambda \oplus \bigoplus_{i \in \mathbb{Z}} \mathcal{A}$

where $\mathcal{A} = \mathcal{D}\Lambda = \text{Hom}_k(\Lambda, k)$ ( $\mathcal{A}$ is a $\Lambda$-$\Lambda$-bimodule)

* The elements of $\hat{\Lambda}$ are denoted by $(a_i, \psi_i)$, where $a_i \in \Lambda$, $\psi_i \in \mathcal{A}$ and almost all of them are zero.

* Product in $\hat{\Lambda}$: $(a_i, \psi_i) \cdot (b_i, \psi_i) = (a_i b_i, a_i + \psi_i b_i)$
The category $\hat{\Lambda}$-mod:

Objects: $\hat{M} = (M_i, f_i)_{i \in \mathbb{Z}}$

$\hat{M}: \ldots \sim M_{i-1} \overset{f_{i-1}}{\sim} M_i \overset{f_i}{\sim} M_{i+1} \sim \ldots$, $M_i \in \mathcal{O}_\Lambda(\Lambda$-mod), almost all zero

$f_i: Q \otimes M_i \rightarrow M_{i+1}$ such that $f_{i+1} \circ (\text{id}_Q \otimes f_i) = 0$

Morphisms: $h: \hat{M} \rightarrow \hat{M}'$

$\hat{M}: \ldots \sim M_{i-1} \overset{f_{i-1}}{\sim} M_i \overset{f_i}{\sim} M_{i+1} \sim \ldots$

$h: \hat{M} \rightarrow \hat{M}'$

$\hat{M}': \ldots \sim M_{i-1} \overset{f_{i-1}}{\sim} M_i \overset{f_i}{\sim} M_{i+1} \sim \ldots$

$\text{id}_Q \otimes h: \Rightarrow \int h_{\Lambda}$

$Q \otimes M_i \overset{f_i}{\rightarrow} M_{i+1}$

$Q \otimes M_i \overset{f_i}{\rightarrow} M_{i+1}$
**A-mod** is a Frobenius category, i.e. it is exact, has enough injective and projective objects and these objects coincide.

* We denote $\hat{\Lambda}\text{-mod}$ the stable category of $\Lambda$, i.e.

$$\text{Ob}(\hat{\Lambda}\text{-mod}) = \text{Ob}(\hat{\Lambda}\text{-mod})$$

and $f, g \in \text{Hom}_{\hat{\Lambda}\text{-mod}}(\mu, \mu') = \text{Hom}_{\hat{\Lambda}}(\mu, \mu')$ are identified if $f - g$ factors through a projective-injective object.

* $\hat{\Lambda}\text{-mod}$ is a $\Delta'$-category (D. Happel, 1991)

with suspension functor $\Omega^{-1}: \hat{\Lambda}\text{-mod} \to \hat{\Lambda}\text{-mod}$
$\mu: \text{mod}(\Lambda) \rightarrow \text{mod}\hat{\Lambda}$

(Happel's Functor)

$\mu$ is an equivalence

$\text{gl.dim } \Lambda < \infty$
An exact triangle comes from a pushout diagram:

\[
\begin{align*}
0 & \to \hat{M} \to I(\hat{M}) \to \Omega^{-1} \hat{M} \to 0 \\
\hat{h} & \downarrow \quad \downarrow \\
0 & \to \hat{M}' \to \hat{M} \to \Omega^{-1} \hat{M} \to 0
\end{align*}
\]

(Conversely, \(\Delta\)'s in \(\hat{\Lambda}\)-mod induce certain short exact sequences in \(\hat{\Lambda}\)-mod (X.W Chen and P. Zhang, 2007).)

Irreducible morphisms in \(\hat{\Lambda}\)-mod: \(\hat{h}: \hat{M} \to \hat{M}'\) is said irreducible in \(\hat{\Lambda}\)-mod if \(\hat{h}\) is neither a split-mono nor a split-epi and if \(\hat{h} = \hat{\nu} \circ \hat{\mu}\), then \(\hat{\mu}\) is split-mono or \(\hat{\nu}\) is split-epi.
Theorem (H. Giraldo, 2018) Assume that $\hat{h}: \hat{M} \rightarrow \hat{M}'$ is irreducible in $\Lambda$-mod.

Then $\hat{h}$ satisfies one of the following:

(i) For all $i \in \mathbb{Z}$, $h_i$ is a split-mono ($\hat{h}$ is smonic).

(ii) For all $i \in \mathbb{Z}$, $h_i$ is a split-epi ($\hat{h}$ is sepic).

(iii) There is a unique $i_0 \in \mathbb{Z}$ such that $h_{i_0}$ is neither a split-mono nor split-epi and $h_{i_0}$ is an irreducible morphism in $\Lambda$-mod. ($\hat{h}$ is simducible).
Auslander–Reiten triangles in \( \hat{\Lambda} \)-mod:

A distinguished triangle \( \hat{\Lambda} \)-modules if the following are satisfied:

\[ \begin{align*}
\text{(ART1)} & \quad \text{The } \hat{\Lambda} \text{-modules } \hat{\Lambda} \text{ and } \hat{\Lambda}^n \text{ are indecomposable;} \\
\text{(ART2)} & \quad \text{The morphism } \hat{\Lambda}^n : \hat{\Lambda} \to \Sigma \hat{\Lambda} \text{ is non-zero;} \\
\text{(ART3)} & \quad \text{If } \hat{\Lambda}^n : \hat{\Lambda} \to \hat{\Lambda}^n \text{ is a morphism in } \hat{\Lambda} \text{-mod which is not split-epic, then there exists } \hat{\Lambda} \text{-mod such that } \hat{\Lambda}^n = \hat{\Lambda} \circ \hat{\Lambda}^n.
\end{align*} \]
Almost split sequences in $\hat{A}$-mod induce AR- $\Delta$'s in $\hat{A}$-mod.

**Question:** How about the converse?

**Theorem (Y. Calderón-Hernán, H. Giardino, JVM, 2022)**

Assume that $\tau: \hat{M} \xrightarrow{\hat{h}} \hat{M}' \xrightarrow{\hat{h}'} \hat{M}'' \xrightarrow{\alpha} \hat{M}$ is an AR $\Delta$ in $\hat{A}$-mod such that neither $\hat{M}$, $\hat{M}'$ nor $\hat{M}''$ has projective direct summands in $\hat{A}$-mod. Then there is an almost split sequence $0 \rightarrow \hat{M} \xrightarrow{(\hat{\alpha})} \hat{M}' \xrightarrow{\hat{\beta}} \hat{M}'' \rightarrow 0$ in $\hat{A}$-mod that induces $\tau$ in $\hat{A}$-mod. If $\hat{P} \neq 0$, then $\hat{P}$ is indecomposable, $\hat{M} \cong \text{rad} \hat{P}$ and $\hat{M}'' = \hat{P}/\text{soc} \hat{P}$. 
Definition: Let $\hat{\mathfrak{h}} : \hat{\mathfrak{M}} \to \hat{\mathfrak{M}}'$ be an irreducible morphism in $\hat{\text{mod}}$. Assume that either $\hat{\mathfrak{M}}$ or $\hat{\mathfrak{M}}'$ is indecomposable and both have no projective direct summands.

(i) $\hat{\mathfrak{h}}$ is stably smonic if $\hat{\mathfrak{h}}$ is smonic in $\hat{\text{mod}}$.
(ii) $\hat{\mathfrak{h}}$ is stably sepic if $\hat{\mathfrak{h}}$ is sepic in $\hat{\text{mod}}$.
(iii) $\hat{\mathfrak{h}}$ is stably sirreducible if $\hat{\mathfrak{h}}$ is sirreducible in $\hat{\text{mod}}$.

Theorem (Y. Caldeón-Henao, H. Giraldo and JHM, 2022) If $\hat{\mathfrak{h}} : \hat{\mathfrak{M}} \to \hat{\mathfrak{M}}'$ is as above then $\hat{\mathfrak{h}}$ is either stably smonic, stably sepic or stably sirreducible.
**Theorem (Y. Calderon-Henao, H. Giraldo and JVM, 2022)**

If \( \hat{M} \xrightarrow{\hat{h}} \hat{M}' \xrightarrow{\hat{h}'} \hat{M}'' \xrightarrow{\hat{h}''} \Omega^{-n} \hat{M} \) is an AR \( \Delta \) in \( \hat{\Lambda}-\text{mod} \) such that neither \( \hat{M} \), \( \hat{M}' \) nor \( \hat{M}'' \) have no projective direct summands in \( \hat{\Lambda}-\text{mod} \). Then:

(i) If \( \hat{h} \) is stably smonic, then \( \hat{h}' \) is stably sepic.

(ii) If \( \hat{h}' \) is stably sepic, then \( \hat{h}'' \) is stably simdivisible.

(iii) If \( \hat{h}'' \) is stably simdivisible, then \( \hat{h}' \) is either stably smonic or stably simdivisible.

**Note:** This result extends that for derived categories of algebras with \( \text{gl.dim}\Lambda \leq 0 \) by (E. Riveiro-Alvarez et al., 2018).
Example:

\[ Q = \chi C \rightarrow z \rightarrow 1 \]

\[ \rho = \frac{1}{2} d \theta, \quad x^2 \gamma, \quad \Lambda = k \partial \gamma \]

\[ g \text{ dim } \Lambda = \infty \]
The diagram shows a series of transformations and operations, with labels indicating steps and outcomes. The text on the right side of the diagram notes:

- \( \hat{b} \) is stably smonic
- \( \hat{b}' \) is stably sepic
- \( \hat{b} \) is stably sepic
- \( \hat{b}' \) is stably smonic and irreducible
The results presented in this talk were published in *Algeb. Repres. Theory* in September, 2022.

Thanks.