A cluster structure on the coordinate ring of partial flag varieties

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Auslander Conference
Oct 26, 2022
Abstract

The main goal is to show that the (multi-homogeneous) coordinate ring of a partial flag variety $\mathbb{C}[G/P_K]$ contains a cluster algebra for any semisimple complex algebraic group $G$. We use derivation properties and a special lifting map to prove that the cluster algebra structure $\mathcal{A}$ of the coordinate ring $\mathbb{C}[N_K]$ of a Schubert cell constructed by Goodearl and Yakimov can be lifted, in an explicit way, to a cluster structure $\hat{\mathcal{A}}$ living in the coordinate ring of the corresponding partial flag variety. Then we use a minimality condition to prove that the cluster algebra $\hat{\mathcal{A}}$ is equal to $\mathbb{C}[G/P_K]$ after localizing some special minors.
Outline

1. Cluster algebra overview
2. Partial flag varieties
3. A cluster structure on $\mathbb{C}[G/P_K]$
Cluster algebra overview

Definition

A seed is a pair $(\tilde{x}, \tilde{B})$ with the following data:

- $\tilde{x}$ is a tuple of algebraically independent variables
  $\tilde{x} = (x_1, ..., x_n, ..., x_m)$;

- $x_1, ..., x_n, ..., x_m$ generate an ambient field $\mathcal{F}$, that is, a field isomorphic to $\mathbb{C}(x_1, ..., x_n, x_{n+1}, ..., x_m)$;

- $\tilde{B}$ is an $m \times n$ matrix whose north $n \times n$ submatrix $B$ is skew-symmetrizable, that is, can be transformed to a skew-symmetric matrix by multiplying each row $r_i$ by some nonzero integer $d_i$;

The tuple $\tilde{x}$ is called an extended cluster, where its first $n$-variables are called the mutable variables and the next $(m - n)$-variables are called the frozen variables. The matrix $\tilde{B}$ is called the exchange matrix.
Definition

Let $(\tilde{x}, \tilde{B})$ be a seed. A mutation $\mu_k$ at $k \in [1, n]$ is a transformation to a new seed $\mu_k(\tilde{x}, \tilde{B}) = (\tilde{x}', \tilde{B}')$, where the entries of the matrix $\tilde{B}'$ are given by

$$
b'_{ij} = \begin{cases} 
-b_{ij}, & \text{if } i = k \text{ or } j = k, \\
b_{ij} + \frac{|b_{ik}| b_{kj} + b_{ik} |b_{kj}|}{2}, & \text{otherwise};
\end{cases}
$$

and $\tilde{x}' = (x'_1, ..., x'_m)$, where $x'_i = x_i$ if $i \neq k$ and

$$x_k x'_k = \prod_{b_{ik} > 0} x_i^{b_{ik}} + \prod_{b_{ik} < 0} x_i^{-b_{ik}}.$$
Remark

It is not hard to verify that $\mu_k$ is an \textit{involution}, that is,

$$\mu_k(\mu_k(\tilde{x}, \tilde{B})) = (\tilde{x}, \tilde{B}).$$

Remark

Let us start with an \textit{initial seed} $(\tilde{x}, \tilde{B})$. It is known that any mutable variable can be obtained from $(\tilde{x}, \tilde{B})$ by some sequence of mutations at some mutable indices. Therefore, knowing an initial seed gives a full picture of the mutable variables, and thus all of the extended clusters.
Definition

Let \((\tilde{x}, \tilde{B})\) be a seed. The \textit{cluster algebra} (of \textit{geometric type}) is the polynomial algebra over \(\mathbb{C}\) of all mutable and frozen variables.
Partial flag varieties

Remark

From now on, the set $I$ denotes the vertex set of the Dynkin diagram $\Delta$ corresponding to $G$.

Definition

A *parabolic* subgroup $P$ of $G$ is a closed subgroup that lies between $G$ and some Borel subgroup $B$. 
Example

1. Any Borel subgroup $B$ is parabolic.

2. Fix a nonempty subset $J \subset I$ and let $K = I \setminus J$. Denote by $x_i(t)$ ($i \in I$, $t \in \mathbb{C}$) the simple root subgroups of the unipotent radical $N$ of $B$ and denote by $y_i(t)$ the simple root subgroups of the unipotent radical $N^-$ of $B^-$. The subgroup $P_K$ generated by $B$ and the one-parameter subgroups $y_k(t)$ ($k \in K$, $t \in \mathbb{C}$) is parabolic. Similarly, the subgroup $P^-_K$ generated by $B^-$ and the one-parameter subgroups $x_k(t)$ ($k \in K$, $t \in \mathbb{C}$) is a parabolic subgroup.
Definition

A quotient $G/P$ is called a \textit{(partial) flag variety} if $P$ is a parabolic subgroup of $G$.

Remark

It is known that any parabolic subgroup is conjugate to a parabolic subgroup of the form $P_K$. This somehow, in many cases, reduces the study of partial flag varieties to the ones of the form $G/P_K$. 
Remark

The partial flag variety $G/P_K^-$ can be naturally embedded as a closed subset of the product of projective spaces

$$\prod_{j \in J} \mathbb{P}(L(\varpi_j)^*),$$

where $\varpi_j$ is a fundamental weight of $G$, and for a dominant weight $\lambda$, the corresponding $L(\lambda)$ is the finite-dimensional irreducible $G$-module with highest weight $\lambda$; and $L(\lambda)^*$ denotes the right $G$-module obtained by twisting the action of $G$. As a terminology, the $L(\varpi_i)$'s are called the fundamental representations.
Remark

Let \( \Pi_J \cong \mathbb{N}^J \) denote the monoid of dominant integral weights of the form \( \lambda = \sum_{j \in J} a_j \varpi_j \), where \( a_j \in \mathbb{N} \). The multi-homogeneous coordinate ring \( \mathbb{C}[G/P^-_K] \) is a \( \Pi_J \)-graded algebra. In particular,

\[
\mathbb{C}[G/P^-_K] = \bigoplus_{\lambda \in \Pi_J} L(\lambda).
\]

One of the significant results is that \( \mathbb{C}[G/P^-_K] \) can be identified with the subalgebra of \( \mathbb{C}[G/N^-] \) generated by the homogeneous elements of degree \( \varpi_j \), where \( j \in J \).
Remark

For a Weyl group $W$ of $G$, the longest element in this paper will always be denoted by $w_0$ and the Coxetor generators will be denoted by $s_i$ where $i$ runs in $I$.

The notation of the length of some $w \in W$ will be $\ell(w)$. The Chevalley generators of the Lie algebra $\mathfrak{g}$ of $G$ are denoted $e_i, f_i, h_i$, where again $i$ runs in $I$. The $e_i$'s here generate $\text{Lie}(N) = \mathfrak{n}$. An important consequence of this is that $N$ acts naturally from the left and right on $\mathbb{C}[N]$ by the following left and right actions respectively:

$$(x \cdot f)(n) = f(nx), \quad (f \in \mathbb{C}[N] \text{ and } x, n \in N),$$

$$(f \cdot x)(n) = f(xn), \quad (f \in \mathbb{C}[N] \text{ and } x, n \in N).$$

One might differentiate these two actions to get left and right actions of $\mathfrak{n}$ on $\mathbb{C}[N]$, respectively.
Notation

The right action of $e_i$ on $f \in \mathbb{C}[\mathcal{N}]$ will be denoted by $e_i^\dagger(f) := f \cdot e_i$.

Remark

Let $G$ be of type $A$. A (flag) minor is a regular irreducible function of $\mathbb{C}[G]$ defined as follows: For each subset $I \subset [1, n] := \{1, \ldots, n\}$ and each matrix $x \in G$, the minor $\Delta_I(x)$ is defined to be the determinant of the submatrix of $x$ whose rows are indexed by $I$ and columns are indexed by $1, \ldots, |I|$. This notion was generalized by Fomin and Zelevinsky to the notion of (generalized) minor $\Delta_{u \varpi_j, w(\varpi_j)}$, where $u, w$ belong to the Weyl group $W$. The notions of flag minors and generalized minors coincide in type $A$. However, the generalized minor notion makes sense in any type.
Remark

There minors $\Delta_{\varpi_j, w(\varpi_j)}$, $w \in W$, are of degree $\varpi_j$. They connect the coordinate ring of the cell with the coordinate ring of the corresponding flag variety by the following rule:

$$\mathbb{C}[N_K] = \mathbb{C}[G/P^-_K]/(\Delta_{\varpi_j, \varpi_j} - 1), \quad (j \in J).$$

Notation

The restriction of non-vanishing $\Delta_{\varpi_j, w(\varpi_j)}$ on $\mathbb{C}[N_K]$ will be denoted by $D_{\varpi_j, w(\varpi_j)}$. 
In the work of Geiß, Leclerc and Schröer, they proved that \( \mathbb{C}[G/P_K^-] \) admits a cluster structure if \( G \) is simply-laced of type \( A_n \) or \( D_4 \), up to certain localization. They conjectured that this is true for any semisimple \( G \). Our work proved this generality with a relaxed localization.
Lemma

For every $f \in \mathbb{C}[N_K]$ there exists a unique homogeneous element $\tilde{f} \in \mathbb{C}[G/P_K^{-}]$ such that its projection to $\mathbb{C}[N_K]$ is $f$ and whose multi-degree is minimal with respect to the usual partial ordering obtained by the usual ordering of weights, that is, $\mu \preceq \lambda$ iff $\lambda - \mu$ is an $\mathbb{N}$-linear combination of weights $\varpi_j$ ($j \in J$).
Remark

The proof of the preceding lemma involves the following important points:

1. The notation $a_j(f)$ means the maximum of $\left\{ s \mid (e^\dagger_j)^sf \neq 0 \right\}$.

2. The notation $\lambda(f)$ means

$$\lambda(f) = \sum_{j \in J} a_j(f) \omega_j.$$

3. The minimality in the previous lemma means that $\lambda(f)$ is minimal in the following sense: if $\tilde{f} \in L(\lambda)$ and $\text{proj}(\tilde{f}) = f$ then $\lambda(f) \leq \lambda$. On the other hand, the projection of each piece $L(\lambda)$ to $\mathbb{C}[N_K]$ is injective and so there if there is an element there whose projection is $f$, then it is unique in $L(\lambda)$. These two pieces of information together are the main ingredients in proving the existence and uniqueness of $\lambda(f)$. 
Remark

One might naively guess that $\tilde{D}_{\varpi l, w(\varpi l)} = \Delta_{\varpi l, w(\varpi l)}$, but this is not true in general.

Our work shows the following:

Theorem (F.K.)

Let $w = s_{i_1}s_{i_2}\ldots s_{i_n} \in \mathcal{W}$ and $\alpha_{i_1}$ be the vertex of the Dynkin diagram indexed by $i_1$.

$$\tilde{D}_{\varpi_{i_k}, w \leq k \varpi_{i_k}} = \frac{\Delta_{\varpi_{i_k}, w \leq k \varpi_{i_k}} \Delta_{d_k \varpi_{i_1}, \varpi_{i_1}}}{\Delta_{\varpi_{i_k}, \varpi_{i_k}}}$$

where $s_{i_1}(s_{i_2}\ldots s_{i_k})(\varpi_{i_n}) = s_{i_2}\ldots s_{i_k}(\varpi_{i_n}) - d_k \alpha_{i_1}$, and $k \in \{1, \ldots, n\}$.
Remark

Indeed, Geiß, Leclerc and Schröer proved that the coordinate ring of a partial flag variety has a cluster structure, for types $A_n$ and $D_4$, by showing the following:

1. The coordinate ring of a Schubert cell has a cluster algebra structure $\mathcal{A}$.

2. The cluster algebra $\mathcal{A}$ of the previous step can be lifted to some special cluster algebra $\hat{\mathcal{A}}$ that lives in the coordinate ring of the partial flag variety corresponding to the coordinate ring of the cell of the previous step.

3. The cluster algebra $\hat{\mathcal{A}}$ is equal to the coordinate ring of the partial flag variety, up to localization by non-minuscule minors indexed by $j \in J$. 
Remark

The program of Geiß, Leclerc and Schröer, can be followed to prove the general argument for any semisimple algebraic complex group $G$. However, unfortunately, some essential tools of the proof work on the simply-laced case only. In fact, it uses some categorification in which works in the simply-laced case only, to show the first and the second steps, while it treated the third step for types $A_n$ and $D_4$ case by case. Because of that, it was not possible for us to use the same tools. Thus, the generalization we seek must use some other results.
However, a significant consequence of the work of Goodearl and Yakimov is the following:

**Theorem**

*For any semisimple complex algebraic group $G$, the coordinate ring $\mathbb{C}[N_K]$ has a canonical cluster algebra structure.*
To see the cluster structure of Goodearl and Yakimov consider the following:

**Definition**

Define the functions $p$ and $s$ by

$$p(k) := \begin{cases} \max\{j < k \mid i_j = i_k\}, & \text{if such } j \text{ exists;} \\ -\infty, & \text{otherwise.} \end{cases}$$

$$s(k) := \begin{cases} \min\{j > k \mid i_j = i_k\}, & \text{if such } j \text{ exists;} \\ \infty, & \text{otherwise.} \end{cases}$$
Remark

Let $i = (i', i'')$ be a word of the longest element $w_0$ of $W$ whose subword $i'$ corresponds to the longest element of $w_0^K$ of the Dynkin subdiagram indexed by $K$. The initial extended cluster variables of Goodearl and Yakimov are given by $D_{\varpi_j, w \leq j \varpi_j}, (j \in I)$, in which the frozen variables are the ones indexed by $j \in I$ such that $s(j) = \infty$. The extended exchange matrix is given by

$$((\widetilde{B}^w)_{jk} = \begin{cases} 
1, & \text{if } j = p(k), \\
-1, & \text{if } j = s(k), \\
a_{ij}i_k, & \text{if } j < k < s(j) < s(k), \\
a_{ij}i_k, & \text{if } k < j < s(k) < s(j), \\
0, & \text{otherwise;}
\end{cases}$$

where the entry $a_{ij}i_k$ is the same $i_j \times i_k$ entry of the Cartan matrix of the same type.
So we have now:

1. The coordinate ring of a Schubert cell has a cluster algebra structure $\mathcal{A}$. ✓

2. The cluster algebra $\mathcal{A}$ of the previous step can be lifted to some special cluster algebra $\hat{\mathcal{A}}$ that lives in the coordinate ring of the partial flag variety corresponding to the coordinate ring of the cell of the previous step.

3. The cluster algebra $\hat{\mathcal{A}}$ is equal to the coordinate ring of the partial flag variety, up to localization by non-minuscule minors indexed by $j \in J$. 
Remember this:

**Lemma**

For every $f \in \mathbb{C}[N_K]$, there exists a unique homogeneous element $\tilde{f} \in \mathbb{C}[G/P^{-}]$ such that its projection to $\mathbb{C}[N_K]$ is $f$ and whose multi-degree is minimal with respect to the usual partial ordering obtained by the usual ordering of weights, that is, $\mu \preceq \lambda$ iff $\lambda - \mu$ is an $\mathbb{N}$-linear combination of weights $\varpi_j$ ($j \in J$).

**Lemma**

For all $f, g \in \mathbb{C}[N_K]$, we have $\tilde{f} \cdot \tilde{g} = \tilde{f} \cdot \tilde{g}$. If for any $j \in J$, $a_j(f + g) = \max\{a_j(f), a_j(g)\}$, then there are some relatively prime monomials $\mu, \nu$ in the generalized minors $\Delta_{\varpi_j, \varpi_j}$ such that

$$\tilde{f} + \tilde{g} = \mu \tilde{f} + \nu \tilde{g}.$$
Remark

Let \((\tilde{x}, \tilde{B})\) be a seed of the cluster algebra \(A = \mathbb{C}[N_K]\). Then the mutation formula tells us that \(x_kx'_k = M_k + L_k\), where \(M_k, L_k\) are monomials in the variables \(x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_n\). As a consequence of the previous lemma we get that

\[
\tilde{x}_k \tilde{x}'_k = \mu_k \tilde{M}_k + \nu_k \tilde{L}_k,
\]

where \(\mu_k\) and \(\nu_k\) are relatively prime monomials in \(\Delta_{\omega_j, \omega_j}\) \((j \in J)\). This means that we can write \(\mu_k\) and \(\nu_k\) as

\[
\mu_k = \prod_{j \in J} \Delta^{\alpha_j}_{\omega_j, \omega_j} \quad \text{and} \quad \nu_k = \prod_{j \in J} \Delta^{\beta_j}_{\omega_j, \omega_j}.
\]
Definition

For any seed \((x, B)\) of the cluster algebra \(A_J = \mathbb{C}[N_K]\) define a new pair \((\hat{x}, \hat{B})\) of \(\mathbb{C}[G/P^\perp_K]\) by raising each variable \(x\) of \((x, B)\) to the variable \(\tilde{x}\) preserving the same type (mutable or frozen) and by adding the generalized minors \(\Delta_{\omega_j, \omega_j}\) modded out in \(\mathbb{C}[N_K]\) as frozen variables. The matrix \(\hat{B}\) of this lift is obtained as follows: Extend the matrix \(B\) of the construction of Goodearl and Yakimov by \(|J|\) rows labeled by the elements of \(J\) such that the entries are

\[
\hat{b}_{jk} = \begin{cases} 
\beta_j, & \text{if } \beta_j \neq 0; \\
-\alpha_j, & \text{else},
\end{cases}
\]

where \(\alpha_j\) and \(\beta_j\) are as in the previous remark.
Theorem (F.K.)

Let \( \{(x, B)\} \) be the collection of seeds of the cluster algebra \( \mathcal{A}_J \) of \( \mathbb{C}[N_K] \). The corresponding collection \( \{\widehat{(x, B)}\} \) constructed above forms a valid collection of seeds. In other words, if \( (x, B) \) and \( (x', B') \) are two seeds of the coordinate ring of the cell \( \mathbb{C}[N_K] \) such that \( (x', B') = \mu_k(x, B) \), then correspondingly \( (\widehat{x}', \widehat{B}') = \mu_k(\widehat{x}, \widehat{B}) \).
Proof idea: For the matrix $\hat{B}$ we use entries from

$$\tilde{x}_k \tilde{x}'_k = \mu_k \tilde{M}_k + \nu_k \tilde{L}_k.$$

So, for the mutation we should show that the mutated matrix entries match the ones coming from

$$\tilde{x}'_t \tilde{x}''_t = \mu'_t \tilde{M}'_t + \nu'_t \tilde{L}'_t.$$

Equivalently, we may assumed that $\mu'_t$ and $\nu'_t$ are as we want and then show that $\mu'_t \tilde{M}'_t + \nu'_t \tilde{L}'_t$, is an element whose proj is $\tilde{M}'_t + \tilde{L}'_t$ and whose order is minimal with respect to $\leq$. This can be done using the derivation properties of the Chevally generator $e_i$. 
Corollary

Let $B$ be the matrix $\tilde{B}^w$ of Goodearl and Yakimov. The pair

$$\left(\{\tilde{D}_{\omega_{ik}, w \leq k \omega_{ik}}\} \sqcup \{\Delta_{\omega_j, \omega_j} \mid j \in J\}, \hat{B}\right)$$

is an initial seed of a cluster algebra $\hat{A} \subset \mathbb{C}[G/P_K]$. 
So we have now:

1. The coordinate ring of a Schubert cell has a cluster algebra structure $\mathcal{A}$. ✓

2. The cluster algebra $\mathcal{A}$ of the previous step can be lifted to some special cluster algebra $\hat{\mathcal{A}}$ that lives in the coordinate ring of the partial flag variety corresponding to the coordinate ring of the cell of the previous step. ✓

3. The cluster algebra $\hat{\mathcal{A}}$ is equal to the coordinate ring of the partial flag variety, up to localization by non-minuscule minors indexed by $j \in J$. 


Theorem (F.K.)

The localization of the homogeneous coordinate ring of the flag variety $\mathbb{C}[G/P_K]$ by $\Delta_{\varpi_j, \varpi_j}, (j \in J)$ equals the localization of the cluster algebra $\hat{A}$ by the same elements. Namely,

\[
\mathbb{C}[G/P_K][\Delta_{\varpi_j, \varpi_j}]_{j \in J} = \hat{A}[\Delta_{\varpi_j, \varpi_j}]_{j \in J}.
\]
Proof idea:

- Take \( f \in \mathbb{C}[G/P_K][\Delta_{\omega_j,\omega_j}]_{j \in J} \) such that \( f \notin \hat{A}[\Delta_{\omega_j,\omega_j}]_{j \in J} \) and it is of minimal degree.
- Obtain a contradiction.
Thus, omitting the non-minuscule restriction on minors, we now get:

1. The coordinate ring of a Schubert cell has a cluster algebra structure $\mathcal{A}$. ✓

2. The cluster algebra $\mathcal{A}$ of the previous step can be lifted to some special cluster algebra $\hat{\mathcal{A}}$ that lives in the coordinate ring of the partial flag variety corresponding to the coordinate ring of the cell of the previous step. ✓

3. The cluster algebra $\hat{\mathcal{A}}$ is equal to the coordinate ring of the partial flag variety, up to localization by minors indexed by $j \in J$. ✓
Example

Let $G$ be a semisimple algebraic group of type $B_3$, say $G = SO_{2(3)+1} = SO_7$, $J = \{3\}$ and $K = I \setminus J = \{1, 2\}$. Consider the longest word

$$w_0 = s_1 s_2 s_1 s_3 s_2 s_1 s_3 s_2 s_3$$

Consider $w = s_3 s_2 s_1 s_3 s_2 s_3$. Since $s(3) = s(5) = s(6) = \infty$ and $s(k) \neq \infty$ for $k \in \{1, 2, 4\}$, we get that the mutable variables are indexed by $1, 2, 4$ and the frozen ones are indexed by $3, 5, 6$ using the function $s$. Therefore, the exchange matrix of the cluster algebra structure of $\mathbb{C}[N_K]$ is
### A cluster structure on $\mathbb{C}[G/P_K]$  

#### Example

\[
\begin{pmatrix}
1 & 2 & 4 \\
0 & a_{i_1 i_2} & 1 \\
-a_{i_2 i_1} & 0 & a_{i_2 i_4} \\
-1 & -a_{i_4 i_2} & 0 \\
-a_{i_3 i_1} & -a_{i_3 i_2} & 0 \\
0 & -1 & -a_{i_5 i_4} \\
0 & 0 & -1
\end{pmatrix}
\]
Example

\[
\begin{pmatrix}
1 & 2 & 4 \\
0 & -2 & 1 \\
1 & 0 & -1 \\
-1 & 2 & 0 \\
0 & 1 & 0 \\
0 & -1 & 1 \\
0 & 0 & -1
\end{pmatrix}
\begin{pmatrix}
1 \\
2 \\
1 \\
4 \\
3 \\
5 \\
6
\end{pmatrix}
\]
A cluster structure on $\mathbb{C}[G/P^-_K]$.

Example

where the column labels denote the cluster variables and the row labels denote the extended cluster variables, as usual. Also, the extended cluster variables $D_{\varpi_{ij}, w \leq j \varpi_{ij}}$ are

<table>
<thead>
<tr>
<th>$j$</th>
<th>$D_{\varpi_{ij}, w \leq j \varpi_{ij}}$</th>
<th>Status</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$D_{\varpi_3, s_3 \varpi_3}$</td>
<td>(mutable)</td>
</tr>
<tr>
<td>2</td>
<td>$D_{\varpi_2, s_3 s_2 \varpi_2}$</td>
<td>(mutable)</td>
</tr>
<tr>
<td>3</td>
<td>$D_{\varpi_1, s_3 s_2 s_1 \varpi_1}$</td>
<td>(frozen)</td>
</tr>
<tr>
<td>4</td>
<td>$D_{\varpi_3, s_3 s_2 s_1 s_3 \varpi_3}$</td>
<td>(mutable)</td>
</tr>
<tr>
<td>5</td>
<td>$D_{\varpi_2, s_3 s_2 s_1 s_3 s_2 \varpi_2}$</td>
<td>(frozen)</td>
</tr>
<tr>
<td>6</td>
<td>$D_{\varpi_3, s_3 s_2 s_1 s_3 s_2 s_3 \varpi_3}$</td>
<td>(frozen)</td>
</tr>
</tbody>
</table>
Example

Therefore, by the main theorem, the following tuple is an initial extended cluster of $\mathbb{C}[G/P_K^-]$

$$\hat{x} = \left( \tilde{D}_{\omega_3, s_3 s_3}, \tilde{D}_{\omega_2, s_3 s_2 s_2}, \tilde{D}_{\omega_3, s_3 s_2 s_1 s_3}, \tilde{D}_{\omega_1, s_3 s_2 s_1 s_1}, \tilde{D}_{\omega_2, s_3 s_2 s_1 s_3 s_2}, \tilde{D}_{\omega_3, s_3 s_2 s_1 s_3 s_2} \right)$$

such that the first three variables are mutable and the rest are frozen. Consequently, the extended exchange matrix $\hat{B}$ of $\mathbb{C}[G/P_K^-]$ attached to this extended cluster is
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\[ \tilde{D}_{\varpi_1,s_3s_2s_1s_2s_1} = \Delta_{s_3s_2s_1s_2s_1}^{\varpi_1,s_1s_2s_1s_2} \]

\[ \tilde{D}_{\varpi_2,s_3s_2s_1s_3s_2} = \Delta_{s_3s_2s_1s_3s_2}^{\varpi_2,s_1s_2s_1s_2} \]

\[ \tilde{D}_{\varpi_3,s_3s_2s_1s_3s_2s_3} = \Delta_{s_3s_2s_1s_3s_2s_3}^{\varpi_3,s_1s_2s_1s_2s_1s_2} \]
Example

\[
\hat{B} = \begin{pmatrix}
1 & 2 & 4 \\
0 & -2 & 1 \\
1 & 0 & -1 \\
-1 & 2 & 0 \\
0 & 1 & 0 \\
0 & -1 & 1 \\
0 & 0 & -1 \\
-1 & 0 & 0
\end{pmatrix} \quad j \in J
\]
Thank you!

A cluster structure on $\mathbb{C}[G/P_K]$