Root of Unity Quantum Cluster Algebras and Cayley–Hamilton Algebras

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MAURICE AUSLANDER
Distinguished Lectures and International Conference 2022
Root of unity quantum cluster algebras

B. Nguyen – K. Trampel – M. Yakimov, 21
H – T. Lê – M. Yakimov, 22
Root of unity based quantum torus

\[ \mathcal{J}_\varepsilon(\Lambda) := k\left[ \varepsilon^{1/2} \right] \langle x_1^\pm, x_2^\pm, \ldots, x_N^\pm \rangle \begin{array}{c} \langle x_j x_k - \varepsilon^{\Lambda(e_j,e_k)} x_k x_j \rangle \end{array} \]

- \( k \): integral domain of characteristic 0.
- \( \varepsilon^{1/2} \): primitive \( \ell \)-th root of unity in the algebraic closure of the fraction field of \( k \).
- \( \{x_1, x_2, \ldots, x_N\} \): cluster variables.
- \( \Lambda : \mathbb{Z}^N \times \mathbb{Z}^N \to \mathbb{Z}/\ell\mathbb{Z} \) is a skew-symmetric bicharacter.
Root of unity based quantum torus

$$\mathcal{T}_\varepsilon(\Lambda) := \frac{\mathbb{k}[\varepsilon^{\frac{1}{2}}]\langle x_1^\pm, x_2^\pm, \ldots, x_N^\pm \rangle}{(x_jx_k - \varepsilon^{\Lambda(e_j, e_k)}x_kx_j)}$$

Root of unity mixed quantum torus

$$\mathcal{T}_\varepsilon(\Lambda) := \frac{\mathbb{k}[\varepsilon^{\frac{1}{2}}]_n[x_1^{\pm 1}, x_m; n \in \text{ex} \cup \text{inv}, m \notin \text{ex} \cup \text{inv}]}{(x_i x_j - \varepsilon^{\Lambda(e_i, e_j)}x_jx_i)}$$

- **ex** $\subseteq [1, N]$: set of indices for exchange variables.
- **[1, N]\ex**: set of indices for frozen variables.
- **inv** $\subseteq [1, N]\ex$: set of the indices for the frozen variables that can be inverted.
\[ \tilde{B} = (b_{ij}) \in M_{N \times \text{ex}}(\mathbb{Z}) : \text{the exchange matrix with a skew-symmetrizable } \text{ex} \times \text{ex} \text{ submatrix.} \]

The seed mutation of \((x, \tilde{B})\) in the direction of \(k \in \text{ex}\),

\[
    x' = \begin{cases} 
    x'_k = \varepsilon_{1}^{n^{l}} x_k^{-1} \prod_{b_{jk} > 0} x_j^{b_{jk}} + \varepsilon_{1}^{n^{l'}} x_k^{-1} \prod_{b_{jk} < 0} x_j^{-b_{jk}} \\
    x'_j = x_j, & \text{if } j \neq k 
    \end{cases}
\]

\[
    \tilde{B}' = (b'_{ij}) = \begin{cases} 
    -b_{ij}, & \text{if } i = k \text{ or } j = k \\
    b_{ij} + \frac{|b_{ik}|b_{kj} + |b_{ik}|b_{kj}|}{2}, & \text{otherwise.} 
    \end{cases}
\]
Mutation equivalence \((x, \tilde{B}) \sim (x', \tilde{B}')\): one seed \((x, \tilde{B})\) is obtained from another seed \((x', \tilde{B}')\) by a sequence of mutations.

Exchange graph: a graph with vertices corresponding to the seeds that are mutation-equivalent to \((x, \tilde{B})\), and edges given by seed mutations.

\(\Theta\): a subset of vertices.

\(\Theta_c\): a connected subset of vertices.
• Root of unity upper quantum cluster algebra

\[ \bigcap_{(x',\tilde{B}') \sim (x,\tilde{B})} \mathcal{T}_\varepsilon(\Lambda') \geq \]

• Partial intersection of mixed quantum tori

\[ U_\varepsilon(\Theta, \text{inv}) := \bigcap_{(x',\tilde{B}') \in \Theta} \mathcal{T}_\varepsilon(\Lambda') \geq \]
Cayley–Hamilton structures

H – T. Lê – M. Yakimov, 22
Definition [Procesi]

Cayley–Hamilton algebra \((R, C, \text{tr})\) of degree \(d\)

- \(C\) is a central subalgebra of \(R\);
- \(i\) is a not a zero divisor of \(R\) for \(1 \leq i \leq d\);
- Trace map \(\text{tr} : R \to C\)

\[
\text{for all } a \in R, \quad \chi_{d,a}(a) = 0, \\
\text{tr}(1) = d.
\]

The \(d\)-th characteristic polynomial \(\chi_{d,a}(t)\) is defined by

\[
\chi_{d,a}(t) := t^d - c_1(a)t^{d-1} + \cdots + (-1)^d c_d(a)
\]

where \(c_i(a) := p_i(\text{tr}(a), \text{tr}(a^2), \ldots, \text{tr}(a^i))\), and \(p_i(\psi_1, \psi_2, \ldots, \psi_i)\) is the \(i\)-th elementary symmetric function where

\[
\psi_i := \lambda_1^i + \lambda_2^i + \cdots + \lambda_d^i
\]

is the Newton power sum function.
Definition [Procesi]

Cayley–Hamilton algebra \((R, C, \text{tr})\) of degree \(d\)

- \(C\) is a central subalgebra of \(R\);
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- Trace map \(\text{tr} : R \rightarrow C\)

for all \(a \in R\), \(\chi_{d,a}(a) = 0\),
\[\text{tr}(1) = d.\]

- Trace map \(\text{tr} : R \rightarrow C\) satisfies

1. \(C\)-linearity: \(\text{tr}(za + wb) = z \text{tr}(a) + w \text{tr}(b), \ \forall a, b \in R, z, w \in C\)
2. \(\text{tr}(ab) = \text{tr}(ba), \ \forall a, b \in R\)
Example

\((M_{r \times r}(C), C, \text{Tr})\) is a Cayley–Hamilton algebra of degree \(r\), where \(C\) is a commutative ring and \(\text{Tr}\) is the matrix trace.

Question

Is the subalgebra of a Cayley–Hamilton algebra also a Cayley–Hamilton algebra?

Lemma 1 [H – Lê – Yakimov, 22]

If \((R, C, \text{tr})\) is a Cayley–Hamilton algebra of degree \(d\) and \(R', C'\) are subalgebras of \(R\), \(R' \cap C\), such that \(\text{tr}(R') \subseteq C'\), then \((R', C', \text{tr}|_{R'})\) is also a Cayley–Hamilton algebra of same degree.
Example of a central subalgebra: full center

$\mathcal{T}_\varepsilon(\Lambda), \mathcal{T}_\varepsilon(\Lambda)_\geq, U_\varepsilon(\Theta, \text{inv})$ have larger centers.

- Denote $x_{e_i} := x_i$ and $x_{e_i+e_j} := \varepsilon \frac{\Lambda(e_i, e_j)}{2} x_{e_i} x_{e_j}$

  $$\implies x_f = \text{some monomial}, \text{ for } f \in \mathbb{Z}^N.$$

- Recall: $x_{e_i} x_{e_j} = \varepsilon^{\Lambda(e_i, e_j)} x_{e_j} x_{e_i} \implies x_f x_g = \varepsilon^{\Lambda(f, g)} x_g x_f$.

- $\text{Ker}(\Lambda) = \{ f \in \mathbb{Z}^N : \Lambda(f, g) = 0 \in \mathbb{Z}/\ell\mathbb{Z}, \forall g \in \mathbb{Z}^N \}$

Examples of full centers

$$\mathcal{Z}(\mathcal{T}_\varepsilon(\Lambda)) = \mathbb{k}[\varepsilon^{\frac{1}{2}}] - \text{Span}\{x_f : f \in \text{Ker}(\Lambda)\}$$

$$\mathcal{Z}(\mathcal{T}_\varepsilon(\Lambda)_\geq) = \mathcal{T}_\varepsilon(\Lambda)_\geq \cap \mathcal{Z}(\mathcal{T}_\varepsilon(\Lambda))$$

$$\mathcal{Z}(U_\varepsilon) = U_\varepsilon \cap \mathcal{Z}(\mathcal{T}_\varepsilon(\Lambda))$$
Example of a central subalgebra:

**Canonical central subalgebra**

- **Observation:** \((x_k)^\ell \in \mathcal{Z}(T_{\epsilon}(\Lambda))\), (recall \(\epsilon^{\ell/2} = 1\))
  
  Denote \(x^\ell = (x_1^\ell, x_2^\ell, \ldots, x_N^\ell)\)

- **Coprime condition:** \(\ell \in \mathbb{Z}_+\) is odd and coprime to the diagonal entries of the skew-symmetrizing matrix \(D\).
  It ensures

\[
(x_k')^\ell = (x_k)^{-\ell} \prod_{b_{jk} > 0} (x_j)^{\ell b_{jk}} + (x_k)^{-\ell} \prod_{b_{jk} < 0} (x_j)^{-\ell b_{jk}}
\]
• Canonical central subalgebras of $T_\varepsilon(\Lambda)$ and $T_\varepsilon(\Lambda)_{\geq}$

\[ T_\varepsilon^\ell(\Lambda) := k[\varepsilon^{1/2}][x_1^{\pm \ell}, x_2^{\pm \ell}, \ldots, x_m^{\pm \ell}, x_{m+1}^{\pm \ell}, \ldots, x_N^{\pm \ell}] \]

\[ T_\varepsilon^\ell(\Lambda)_{\geq} := k[\varepsilon^{1/2}][x_1^{\pm \ell}, x_2^{\pm \ell}, \ldots, x_m^{\pm \ell}, x_{m+1}^{\ell}, \ldots, x_N^{\ell}] \]

• Canonical central subalgebra of $U_\varepsilon(\Theta, \text{inv})$

\[ CU_\varepsilon(\Theta, \text{inv}) := \bigcap_{(x', \tilde{B}') \in \Theta} T_\varepsilon^\ell(\Lambda')_{\geq} \]

Proposition [H – Lê – Yakimov, 22]

If the Coprime condition is satisfied, then $CU_\varepsilon(\Theta, \text{inv}) \cong U(\Theta, \text{inv})_{k[\varepsilon^{1/2}]}$, where $U(\Theta, \text{inv})_{k[\varepsilon^{1/2}]}$ is the classical upper cluster algebra over $k[\varepsilon^{1/2}]$.

Note: for later discussion, we assume the Coprime condition is satisfied.
Example of a trace map: regular trace

Assume

(1) $R$ is a $k$-algebra over a commutative ring $k$,
(2) $R$ is a free module over a central subalgebra $C$ of finite rank $r = [R : C]$,

the regular trace $\text{tr}_{\text{reg}} : R \to C$ is given by

$$
\text{tr}_{\text{reg}} : R \to \text{End}_C(R) \cong M_r(C) \xrightarrow{\text{Tr}} C
$$

The first map is given by the left action of $R$ on itself.
Example of a trace map: reduced trace

- Assume

  1. $R$ is a prime PI algebra of PI degree $d$,
  2. $Q = \text{Frac}(\mathcal{Z}(R))$,

the reduced trace $\text{tr}_{\text{red}} : R \to \mathcal{Z}(R)$ is given by

$$\text{tr}_{\text{red}} : R \to R \otimes_{\mathcal{Z}(R)} Q \cong M_t(D) \to M_d(F) \xrightarrow{\text{Tr}} F$$

By Posner’s theorem, $R \otimes_{\mathcal{Z}(R)} Q$ is a central simple algebra. For the second map (isomorphism), $D$ is a skew field with center $Q$. And $F$ is the maximal subfield of $D$ that contains $Q$. 
Special case: If $R$ is a prime PI algebra of PI degree $d$, free over the center of rank $r$, then $d = \sqrt{r}$ and $\text{tr}_{\text{reg}} = d \text{tr}_{\text{red}}$.

Determine Cayley-Hamilton structures of $T_{\varepsilon}(\Lambda), T_{\varepsilon}(\Lambda)_{\geq}, U_{\varepsilon}(\Theta, \text{inv})$: start with the simplest one using known trace maps (regular trace or reduced trace), then construct the trace maps for other algebras.
Examples of Cayley-Hamilton algebras
– with canonical central subalgebras

\[(\mathcal{T}_\varepsilon(\Lambda)_{\geq}, \mathcal{T}_\varepsilon(\Lambda)_\ell, \text{tr}_{\text{reg}}^{\mathcal{T}_\varepsilon(\Lambda)_{\geq}}) \quad \text{and} \quad (\mathcal{T}_\varepsilon(\Lambda), \mathcal{T}_\varepsilon(\Lambda)_\ell, \text{tr}_{\text{reg}}^{\mathcal{T}_\varepsilon(\Lambda)})\]

- $\mathcal{T}_\varepsilon(\Lambda)_{\geq}$ is a free $\mathcal{T}_\varepsilon(\Lambda)_\ell$-module of rank $\ell^N$.

\[
\text{tr}_{\text{reg}}^{\mathcal{T}_\varepsilon(\Lambda)_{\geq}}(x_f) = \begin{cases} 
\ell^N \cdot x_f, & \text{if } f \in (\ell\mathbb{Z})^N \\
0, & \text{if } f \notin (\ell\mathbb{Z})^N.
\end{cases}
\]

Remark:
- These are the simplest cases with regular traces.
- $U_\varepsilon(\Theta, \text{inv})$ is the subalgebra of $\mathcal{T}_\varepsilon(\Lambda)_{\geq}$, we consider using Lemma 1.
Examples of Cayley-Hamilton algebras
– with canonical central subalgebras

\[(U_\delta, CU_\delta, \text{tr}_{\text{reg}}^T(\Lambda) \geq |U_\delta|).\]

- \(\text{tr}_{\text{reg}}^T(\Lambda) \geq |U_\delta|\) is independent of the seed chosen, but \(U_\delta\) is over a connected set of vertices, i.e., \(U_\delta(\Theta_c, \text{inv})\).

- \(U_\delta\) is usually not free over \(CU_\delta\), so \(\text{tr}_{\text{reg}}^T(\Lambda) \geq |U_\delta|\) is usually not a regular trace.
Examples of Cayley-Hamilton algebras – with full centers

\((\mathcal{T}_\varepsilon(\Lambda), \mathbb{Z}(\mathcal{T}_\varepsilon(\Lambda)), \text{tr}_{\text{red}}^{\mathcal{T}_\varepsilon(\Lambda)})\)

- \(\mathcal{T}_\varepsilon(\Lambda)\) is free over \(\mathbb{Z}(\mathcal{T}_\varepsilon(\Lambda))\) of rank \(r = [\mathbb{Z}^N : \text{Ker}(\Lambda)]\).
- \(\mathcal{T}_\varepsilon(\Lambda)\) is a prime PI algebra of PI degree \(d = \sqrt{[\mathbb{Z}^N : \text{Ker}(\Lambda)]}\).
- \(\text{tr}_{\text{reg}} = d \text{tr}_{\text{red}}\).

Remark:

- This is the simplest case with a reduced trace.
- \(\mathcal{T}_\varepsilon(\Lambda)_{\geq}, \mathcal{U}_\varepsilon(\Theta, \text{inv})\) are the subalgebras of \(\mathcal{T}_\varepsilon(\Lambda)\), we consider using Lemma 1.
Examples of Cayley-Hamilton algebras – with full centers

\((\mathcal{T}_\varepsilon(\Lambda) \geq, \mathcal{Z}(\mathcal{T}_\varepsilon(\Lambda) \geq), \text{tr}_{\text{red}}^{\mathcal{T}_\varepsilon(\Lambda)}|_{\mathcal{T}_\varepsilon(\Lambda) \geq})\)

- \(\text{tr}_{\text{red}}^{\mathcal{T}_\varepsilon(\Lambda)}|_{\mathcal{T}_\varepsilon(\Lambda) \geq}\) is the reduced trace map defined over \(\mathcal{T}_\varepsilon(\Lambda)\) restricted to \(\mathcal{T}_\varepsilon(\Lambda) \geq\).

- It is actually the reduced trace for \(\mathcal{T}_\varepsilon(\Lambda) \geq\).

\((\mathcal{U}_\varepsilon, \mathcal{Z}(\mathcal{U}_\varepsilon), \text{tr}_{\text{red}}^{\mathcal{T}_\varepsilon(\Lambda)}|_{\mathcal{U}_\varepsilon})\)

- \(\text{tr}_{\text{red}}^{\mathcal{T}_\varepsilon(\Lambda)}|_{\mathcal{U}_\varepsilon}\) is independent of the seed chosen.

- It is actually the reduced trace of \(\mathcal{U}_\varepsilon\).
Maximal orders

Every maximal order in a central simple algebra of PI degree $d$ whose center has characteristic $p \not\in [1, d]$ is a Cayley-Hamilton algebra of degree $d$ with respect to its full center and reduced trace.

- Examples: $T_\varepsilon(\Lambda), T_\varepsilon(\Lambda)_\geq$ are maximal orders.
- Question: Is $U_\varepsilon(\Theta, \text{inv})$ also a maximal order? Yes.

**Lemma 2 [H – Lê – Yakimov, 22]**

Assume that $\{R_\gamma \mid \gamma \in \Gamma\}$ is a collection of $k$-algebras over a commutative ring $k$, which are maximal orders in a central simple algebra $S$ for an index set $\Gamma$.

If $R_\gamma$ is a central localization of $R := \cap_\gamma R_\gamma$ then $R$ is a maximal order in $S$.

- $T_\varepsilon(\Lambda')_\geq$ is a central localization of $U_\varepsilon(\Theta, \text{inv})$ for any $(x', \tilde{B}') \in \Theta$. 
Monomial Subalgebras

For a seed $\Sigma = (x, \tilde{B})$, the monomial subalgebra $A_\Sigma$ of $T_\varepsilon(\Lambda)$ is

$$A_\Sigma := k[\varepsilon^{1/2}] - \text{Span}\{x_f \mid f \in \Phi_\Sigma\}$$

where $\Phi_\Sigma \subseteq \mathbb{Z}^N$ is a submonoid.

$\quad \triangleright \quad T_\varepsilon(\Lambda), T_\varepsilon(\Lambda)_\geq$ are special examples of $A_\Sigma$.

$$T_\varepsilon(\Lambda) = k[\varepsilon^{1/2}] - \text{Span}\{x_f \mid f \in \mathbb{Z}^N\}$$

Partial intersection of monomial subalgebras

$$A := \bigcap_{\Sigma \in \Theta} A_\Sigma$$

$\quad \triangleright \quad U_\varepsilon(\Theta, \text{inv})$ is a special example of $A$. 
Monomial Subalgebras

- \((A_\Sigma, \mathcal{Z}(A_\Sigma), \text{tr}_{\text{red}}^\mathcal{T}_\varepsilon(\Lambda) |_{A_\Sigma})\) and \((A, \bigcap_{\Sigma \in \Theta} \mathcal{Z}(A_\Sigma), \text{tr}_{\text{red}}^\mathcal{T}_\varepsilon(\Lambda) |_A)\) are Cayley-Hamilton algebras.

- In general, \(A_\Sigma\) and \(A\) are not maximal orders.
Thank you