Cluster Structures of Double Bott-Samelson Cells

Daping Weng

Michigan State University

April 2019

Joint work with Linhui Shen

arXiv:1904.07992
Motivation: Bott-Samelson Variety

Let $G, B, W$ be defined as usual. Let $\mathbf{i} = (i_1, \ldots, i_l)$ be a reduced word of $w$. The Bott-Samelson variety associated to the reduced word $\mathbf{i}$ is

$$\prod_{i_1} \times \prod_{i_2} \times \ldots \times \prod_{i_l} \bigg/ B$$

where $\prod_i = B \sqcup Bs_iB$. 

Note that

$$\prod_{i_1} \times \prod_{i_2} \times \ldots \times \prod_{i_l} = \bigoplus_j \prod_{j_1} \times \ldots \times \prod_{j_m}$$

where $j = (j_1, \ldots, j_m)$ runs over all subwords of $i$ (not necessarily reduced). These can be thought of as "Bott-Samelson cells". Alternatively one can think of an element of $\prod_{j_1} \times \ldots \times \prod_{j_m}$ as a sequence of flags that satisfies the relative position conditions imposed by the simple reflections $s_{j_1}, s_{j_2}, \ldots, s_{j_m}$. So a "double Bott-Samelson cell" will then be two sequences of flags that satisfy two sequences of relative position conditions imposed by two words $i$ and $j$. 
Motivation: Bott-Samelson Variety

- Let $G$, $B$, $W$ be defined as usual. Let $i = (i_1, \ldots, i_l)$ be a reduced word of $w$. The Bott-Samelson variety associated to the reduced word $i$ is

$$P_{i_1} \times B \left/ \right. P_{i_2} \times \ldots \times P_{i_l} / B$$

where $P_i = B \sqcup Bs_iB$.

- Note that

$$P_{i_1} \times \ldots \times P_{i_l} = \bigsqcup_{j \subseteq i} (Bs_{j_1}B) \times \ldots \times (Bs_{j_m}B)$$

where $j = (j_1, \ldots, j_m)$ runs over all subwords of $i$ (not necessarily reduced). These can be thought of as “Bott-Samelson cell”.
Motivation: Bott-Samelson Variety

- Let $G$, $B$, $W$ be defined as usual. Let $i = (i_1, \ldots, i_l)$ be a reduced word of $w$. The Bott-Samelson variety associated to the reduced word $i$ is

$$\frac{P_{i_1} \times P_{i_2} \times \ldots \times P_{i_l}}{B}$$

where $P_i = B \sqcup B s_i B$.

- Note that

$$P_{i_1} \times \ldots \times P_{i_l} = \bigsqcup_{j \subset i} (B s_{j_1} B) \times \ldots \times (B s_{j_m} B)$$

where $j = (j_1, \ldots, j_m)$ runs over all subwords of $i$ (not necessarily reduced). These can be thought of as “Bott-Samelson cell”.

- Alternatively one can think of an element of $(B s_{j_1} B) \times \ldots \times (B s_{j_m} B)$ as a sequence of flags that satisfies the relative position conditions imposed by the simple reflections $s_{j_1}, s_{j_2}, \ldots, s_{j_m}$. So a “double Bott-Samelson cell” will then be two sequences of flags that satisfy two sequences of relative position conditions imposed by two words $i$ and $j$. 
Definition

Let $G$ be the Kac-Peterson group (the smallest Kac-Moody group) associated to a symmetrizable generalized Cartan matrix and let $B_{\pm}$ be the two opposite Borel subgroups.
Definition

- Let $G$ be the Kac-Peterson group (the smallest Kac-Moody group) associated to a symmetrizable generalized Cartan matrix and let $B_{\pm}$ be the two opposite Borel subgroups.
- Let $B_{\pm} = \{\text{Borel subgroups that are conjugates of } B_{\pm}\}$. Bruhat decomposition implies that the $G$-orbits in $B_{+} \times B_{+}$ and $B_{-} \times B_{-}$ are parametrized by the Weyl group $W$. 

Notation

- We use superscript to denote Borel subgroups in $B_{+}$, e.g. $B_{0}^{+}$, $B_{1}^{+}$, etc.
- We use subscript to denote Borel subgroups in $B_{-}$, e.g. $B_{0}^{-}$, $B_{1}^{-}$, etc.
- We write $B_{0}^{+} \xrightarrow{\cdot \cdot} B_{1}^{+}$ if $(B_{0}^{+}, B_{1}^{+})$ is in the $w$-orbit in $B_{+} \times B_{+}$.
- We write $B_{0}^{-} \xrightarrow{\cdot \cdot} B_{1}^{-}$ if $(B_{0}^{-}, B_{1}^{-})$ is in the $w$-orbit in $B_{-} \times B_{-}$.
- We write $B_{0}^{+} = B_{0}^{-}$ if $(B_{0}^{+}, B_{0}^{-}) = (gB_{-}, gB_{+})$ for some $g \in G$. 

$B_{+}/B_{+}$ can be thought of as the moduli space of $B_{1}$ satisfying $B_{0}^{+} \xrightarrow{\cdot \cdot} B_{1}^{+}$ for a fixed $B_{0}^{+}$. 
Definition

Let $G$ be the Kac-Peterson group (the smallest Kac-Moody group) associated to a symmetrizable generalized Cartan matrix and let $B_{\pm}$ be the two opposite Borel subgroups.

Let $B_{\pm} = \{\text{Borel subgroups that are conjugates of } B_{\pm}\}$. Bruhat decomposition implies that the $G$-orbits in $B_+ \times B_+$ and $B_- \times B_-$ are parametrized by the Weyl group $W$.

Notation

- We use superscript to denote Borel subgroups in $B_+$, e.g. $B^0$, $B^1$, etc.
- We use subscript to denote Borel subgroups in $B_-$, e.g. $B_0$, $B_1$, etc.
- We write $B^0 \xrightarrow{w} B^1$ if $(B^0, B^1)$ is in the $w$-orbit in $B_+ \times B_+$.
- We write $B_0 \xrightarrow{w} B_1$ if $(B_0, B_1)$ is in the $w$-orbit in $B_- \times B_-.$
- We write $B_0 \xrightarrow{w} B^0$ if $(B_0, B^0) = (gB_-, gB_+)$ for some $g \in G$. 

Cluster Structures of Double Bott-Samelson Cells
Definition

- Let $G$ be the Kac-Peterson group (the smallest Kac-Moody group) associated to a symmetrizable generalized Cartan matrix and let $B_{\pm}$ be the two opposite Borel subgroups.
- Let $B_{\pm} = \{\text{Borel subgroups that are conjugates of } B_{\pm}\}$. Bruhat decomposition implies that the $G$-orbits in $B_+ \times B_+$ and $B_- \times B_-$ are parametrized by the Weyl group $W$.

Notation

- We use superscript to denote Borel subgroups in $B_+$, e.g. $B^0$, $B^1$, etc.
- We use subscript to denote Borel subgroups in $B_-$, e.g. $B_0$, $B_1$, etc.
- We write $B^0 \xrightarrow{w} B^1$ if $(B^0, B^1)$ is in the $w$-orbit in $B_+ \times B_+$.
- We write $B_0 \xrightarrow{w} B_1$ if $(B_0, B_1)$ is in the $w$-orbit in $B_- \times B_-$. 
- We write $B_0 \xrightarrow{s_i} B^0$ if $(B_0, B^0) = (gB_-, gB_+)$ for some $g \in G$.

- $B_+ s_i B_+ / B_+$ can be thought of as the moduli space of $B_1$ satisfying $B_0 \xrightarrow{s_i} B_1$ for a fixed $B_0$. 
Definition

Let $b$ and $d$ be two positive braids in the associated braid group. First choose a word $(i_1, i_2, \ldots, i_m)$ for $b$ and a word $(j_1, j_2, \ldots, j_n)$ for $d$. The *undecorated double Bott-Samelson cell* $\text{Conf}_d^b(B)$ is defined to be

\[
\left\{ \begin{array}{c}
\begin{array}{c}
B^0 \xrightarrow{s_{i_1}} B^1 \xrightarrow{s_{i_2}} \cdots \xrightarrow{s_{i_m}} B^m \\
B_0 \xrightarrow{s_{j_1}} B_1 \xrightarrow{s_{j_2}} \cdots \xrightarrow{s_{j_n}} B_n
\end{array}
\end{array} \right\} / G
\]
Definition

Let $b$ and $d$ be two positive braids in the associated braid group. First choose a word $(i_1, i_2, \ldots, i_m)$ for $b$ and a word $(j_1, j_2, \ldots, j_n)$ for $d$. The undecorated double Bott-Samelson cell $\text{Conf}^b_d(B)$ is defined to be

\[
\begin{align*}
\left\{ \begin{array}{c}
B^0 \xrightarrow{s_{i_1}} B^1 \xrightarrow{s_{i_2}} \cdots \xrightarrow{s_{i_m}} B^m \\
B_0 \xrightarrow{s_{j_1}} B_1 \xrightarrow{s_{j_2}} \cdots \xrightarrow{s_{j_n}} B_n
\end{array} \right\}/G
\end{align*}
\]

Remark

The resulting space does not depend on the choice of words for $b$ and $d$. 
Definition

Let $U_{\pm} := [B_{\pm}, B_{\pm}]$ and define *decorated flag varieties* $A_{\pm} := G/U_{\pm}$. We denote decorated flags with a symbol $A$ instead of $B$.

Definition

The *decorated double Bott-Samelson cell* $\text{Conf}^b_d(A)$ is defined to be

\[
\left\{ \begin{array}{c}
A^0 \xrightarrow{s_{i_1}} B^1 \xrightarrow{s_{i_2}} \ldots \xrightarrow{s_{i_m}} B^m \\
| \\
B_0 \xrightarrow{s_{j_1}} B_1 \xrightarrow{s_{j_2}} \ldots \xrightarrow{s_{j_n}} A_n
\end{array} \right\} / G
\]
Definition

Let $U_\pm := [B_\pm, B_\pm]$ and define decorated flag varieties $A_\pm := G/U_\pm$. We denote decorated flags with a symbol $A$ instead of $B$.

Definition

The decorated double Bott-Samelson cell $\text{Conf}_d^b(A)$ is defined to be

$$
\begin{cases}
A^0 \xrightarrow{s_{i_1}} B^1 \xrightarrow{s_{i_2}} \ldots \xrightarrow{s_{i_m}} B^m \\
B_0 \xrightarrow{s_{j_1}} B_1 \xrightarrow{s_{j_2}} \ldots \xrightarrow{s_{j_n}} A_n
\end{cases}
\bigg/ G
$$

Decorated double Bott-Samelson cell can be viewed as a generalization of double Bruhat cells $B_+ u B_+ \cap B_- v B_-$. Double Bruhat cells are examples of cluster varieties and are studied by Berenstein, Fomin, and Zelevinsky [BFZ05], Fock and Goncharov [FG06], and many others.
Definition

Let $U_{\pm} : = [B_{\pm}, B_{\pm}]$ and define \textit{decorated flag varieties} $A_{\pm} : = G / U_{\pm}$. We denote decorated flags with a symbol $A$ instead of $B$.

Definition

The \textit{decorated double Bott-Samelson cell} $Conf^{d}_{b}(A)$ is defined to be

\[
\begin{align*}
\left\{ \begin{array}{c}
A^0 \xrightarrow{s_{i_1}} B^1 \xrightarrow{s_{i_2}} \ldots \xrightarrow{s_{i_m}} B^m \\
B_0 \xrightarrow{s_{j_1}} B_1 \xrightarrow{s_{j_2}} \ldots \xrightarrow{s_{j_n}} A_n
\end{array} \right\} / G
\end{align*}
\]

Decorated double Bott-Samelson cell can be viewed as a generalization of double Bruhat cells $B_{+} u B_{+} \cap B_{-} v B_{-}$. Double Bruhat cells are examples of cluster varieties and are studied by Berenstein, Fomin, and Zelevinsky [BFZ05], Fock and Goncharov [FG06], and many others.

Theorem (Shen-W.)

\textit{The decorated double Bott-Samelson cells} $Conf^{d}_{b}(A)$ \textit{are smooth affine varieties}. 
Cluster Structures

- We equip each double Bott-Samelson cell (both undecorated and decorated) with an atlas of algebraic torus charts, parametrized by a choice of words for $b$ and $d$ and a triangulation of the “trapezoid”.

```
  B^0 \rightarrow B^1 \rightarrow B^2 \rightarrow B^3

  B_0 \quad B_1 \quad B_2 \quad B_3 \quad B_4 \quad B_5
  \quad \quad s_2 \quad s_1 \quad s_2 \quad s_3 \quad s_3
```
We equip each double Bott-Samelson cell (both undecorated and decorated) with an atlas of algebraic torus charts, parametrized by a choice of words for $b$ and $d$ and a triangulation of the "trapezoid".

There are two kinds of moves available to us:

- Diagonal flipping
- Braid move
Cluster Structures

- We actually consider two versions of decorated double Bott-Samelson cells, one for \( G_{sc} \) and one for \( G_{ad} \) (analogues of the simply-connected form and the adjoint form in the semisimple cases).
Cluster Structures

- We actually consider two versions of decorated double Bott-Samelson cells, one for $G_{sc}$ and one for $G_{ad}$ (analogues of the simply-connected form and the adjoint form in the semisimple cases).

- The natural projection $G_{sc} \rightarrow G_{ad}$ gives rise to natural projection maps $A_{sc} \rightarrow A_{ad}$ and $p : \text{Conf}_d^b (A_{sc}) \rightarrow \text{Conf}_d^b (A_{ad})$. 
We actually consider two versions of decorated double Bott-Samelson cells, one for $G_{sc}$ and one for $G_{ad}$ (analogues of the simply-connected form and the adjoint form in the semisimple cases).

The natural projection $G_{sc} \rightarrow G_{ad}$ gives rise to natural projection maps $\mathcal{A}_{sc} \rightarrow \mathcal{A}_{ad}$ and $p : \text{Conf}^b_d(\mathcal{A}_{sc}) \rightarrow \text{Conf}^b_d(\mathcal{A}_{ad})$.

**Theorem (Shen-W.)**

*The atlas of algebraic torus charts are related by birational maps called cluster mutations. These charts equips $\mathcal{O}(\text{Conf}^b_d(\mathcal{A}_{sc}))$ with the structure of an upper cluster algebra, and equips $\mathcal{O}(\text{Conf}^b_d(\mathcal{A}_{ad}))$ with the structure of an upper cluster Poisson algebra. The pair $(\text{Conf}^b_d(\mathcal{A}_{sc}), \text{Conf}^b_d(\mathcal{A}_{ad}))$ form a Fock-Goncharov cluster ensemble.*
We constructed biregular maps called *reflection maps*:

$$\text{Conf}_{d}^{bs_{i}}(\mathcal{B}) \leftrightarrow \text{Conf}_{ds_{i}}^{b}(\mathcal{B}) \hspace{1cm} \text{Conf}_{d}^{s_{i}b}(\mathcal{B}) \leftrightarrow \text{Conf}_{s_{i}d}^{b}(\mathcal{B}).$$

They are induced by moves that look like the following:

$$B^{0} \xrightarrow{s_{i}} B^{1} \hspace{1cm} B^{0} \xleftarrow{s_{i}} B^{1}$$

These reflection maps are Poisson and respect the cluster structures. One can think of such reflection maps as movement of tangles in a link.
Reflection Maps between double Bott-Samelson cells

- We constructed biregular maps called *reflection maps*:

\[
\text{Conf}_{d}^{bs_{i}}(\mathcal{B}) \leftrightarrow \text{Conf}_{ds_{i}}^{b}(\mathcal{B}) \quad \text{Conf}_{d}^{s_{i}b}(\mathcal{B}) \leftrightarrow \text{Conf}_{s_{i}d}^{b}(\mathcal{B}).
\]

They are induced by moves that look like the following:

\[
\begin{align*}
B_{0}^{0} & \xrightarrow{s_{i}} B_{1}^{1} \\
& \quad \leftrightarrow \\
& \quad B_{0}^{1} \xleftarrow{s_{i}} B_{1}^{0}
\end{align*}
\]

- These reflection maps are Poisson and respect the cluster structures.
Reflection Maps between double Bott-Samelson cells

- We constructed biregular maps called *reflection maps*:

$$\text{Conf}^{bs_i}_d(B) \leftrightarrow \text{Conf}^b_{ds_i}(B) \quad \text{Conf}^{s_i b}_d(B) \leftrightarrow \text{Conf}^b_{s_i d}(B).$$

They are induced by moves that look like the following:

$$B^0 \xrightarrow{s_i} B^1 \quad B^0 \leftrightarrow B^0 \quad B^0 \xrightarrow{s_i} B^1$$

- These reflection maps are Poisson and respect the cluster structures.
- One can think of such reflection maps as movement of tangles in a link.

$$\text{Conf}^{s_1 s_2}_{s_1}(B) \leftrightarrow \text{Conf}^{s_1 s_2}_{s_1}(B)$$
Cluster Donaldson-Thomas Transformation

One important conjecture in cluster theory is the Fock-Goncharov cluster duality [FG09], which conjectures the existence of canonical bases in an upper cluster algebra and its corresponding upper cluster Poisson algebra.
One important conjecture in cluster theory is the Fock-Goncharov cluster duality [FG09], which conjectures the existence of canonical bases in an upper cluster algebra and its corresponding upper cluster Poisson algebra.

Part of a sufficient condition [GHKK18] [GS18] of the duality conjecture is the existence of the cluster Donaldson-Thomas transformation.
One important conjecture in cluster theory is the Fock-Goncharov cluster duality \cite{FG09}, which conjectures the existence of canonical bases in an upper cluster algebra and its corresponding upper cluster Poisson algebra.

Part of a sufficient condition \cite{GHKK18} \cite{GS18} of the duality conjecture is the existence of the cluster Donaldson-Thomas transformation.

**Theorem (Shen-W.)**

Cluster Donaldson-Thomas transformations exist on double Bott-Samelson cells and are given by compositions of reflection maps and a transposition map. Reflection maps intertwine the cluster Donaldson-Thomas transformations on different double Bott-Samelson cells. By verifying the sufficient condition, we prove the cluster duality conjecture for double Bott-Samelson cells.
For the rest of the talk, let $G$ be semisimple and let $w_0$ denote the longest Weyl group element.
For the rest of the talk, let $G$ be semisimple and let $w_0$ denote the longest Weyl group element.

**Theorem (Shen-W.)**

Let $G$ be a semisimple group. Let $b$ be a positive braid and let $m, n$ be two positive integers such that $b^m = w_0^{2n}$. Then the order of the cluster Donaldson-Thomas transformation of $\text{Conf}_b^e(B)$ is finite and divides $2(m + n)$.
For the rest of the talk, let $G$ be semisimple and let $w_0$ denote the longest Weyl group element.

**Theorem (Shen-W.)**

Let $G$ be a semisimple group. Let $b$ be a positive braid and let $m, n$ be two positive integers such that $b^m = w_0^{2n}$. Then the order of the cluster Donaldson-Thomas transformation of $\text{Conf}^e_b(B)$ is finite and divides $2(m + n)$.

**Example**

Suppose $G = SL_3$ and $b = s_1 s_2 s_1 s_2$. Then $b^3 = w_0^4$ in the braid group, and therefore $\text{DT}^{10} = \text{Id}$ on $\text{Conf}^e_b(B)$. Intertwining by a reflection map, this computation also implies that $\text{DT}^{10} = \text{Id}$ on $\text{Conf}^{s_1}_{w_0}(B)$ in the double Bruhat cell case as well.
One version of the conjecture (formulated by Keller [Kel13]) is about the periodicities of the Donaldson-Thomas transformations associated to products of two Dynkin diagrams.

**Theorem (Keller)**

Let $D$ and $D'$ be Dynkin quivers with Coxeter numbers $h$ and $h'$. Then

$$DT_{D \boxtimes D'}^{2(h+h')} = Id.$$
New Proof of Zamolodchikov’s Periodicity Conjecture

- One version of the conjecture (formulated by Keller [Kel13]) is about the periodicities of the Donaldson-Thomas transformations associated to products of two Dynkin diagrams.

Theorem (Keller)

Let $D$ and $D'$ be Dynkin quivers with Coxeter numbers $h$ and $h'$. Then

\[ \text{DT}^{2(h+h')}_{D \boxtimes D'} = \text{Id}. \]

- Using our result on the periodicity of $\text{DT}$ on double Bott-Samelson cells, we can give a new geometric proof of the periodicity conjecture in the case of $D \boxtimes A_n$. 
New Proof of Zamolodchikov’s Periodicity Conjecture

One version of the conjecture (formulated by Keller [Kel13]) is about the periodicities of the Donaldson-Thomas transformations associated to products of two Dynkin diagrams.

Theorem (Keller)

Let $D$ and $D'$ be Dynkin quivers with Coxeter numbers $h$ and $h'$. Then

$$DT_{D \boxtimes D'}^{2(h+h')} = \text{Id}.$$ 

Using our result on the periodicity of $DT$ on double Bott-Samelson cells, we can give a new geometric proof of the periodicity conjecture in the case of $D \boxtimes A_n$.

Give $D$ a bipartite coloring.

$$b = s_2 s_4 s_5, \quad w = s_1 s_3$$
Consider the double Bott-Samelson cell $\text{Conf}_{bwb...}(B)$, where the number of $b$ and $w$ in each braid sum up to $n + 1$. 

Let $h$ be the Coxeter number of $D$. Since $(c^n+1)h = w^2(n+1)$, our result implies that $DT_2(h+n+1)D \triangleright A_n = \text{Id}$. 

\[ 
\begin{array}{c}
\bullet \leftrightarrow \circ \rightarrow \bullet \\
\uparrow \downarrow \uparrow \downarrow \\
\bullet \leftarrow \circ \rightarrow \bullet \\
\downarrow \uparrow \downarrow \uparrow \\
\bullet \leftarrow \circ \rightarrow \bullet \\
\end{array} 
\]
Consider the double Bott-Samelson cell $\text{Conf}_{wbw\ldots}^{bw}(B)$, where the number of $b$ and $w$ in each braid sum up to $n + 1$.

Note that $bw = c$ and $\text{Conf}_{wbw\ldots}^{bw}(B) \cong \text{Conf}_{c^{n+1}}^{e}(B)$. 
Consider the double Bott-Samelson cell $\text{Conf}^{bw\ldots}_{wbw\ldots}(\mathcal{B})$, where the number of $b$ and $w$ in each braid sum up to $n + 1$.

Note that $bw = c$ and $\text{Conf}^{bw\ldots}_{wbw\ldots}(\mathcal{B}) \cong \text{Conf}^e_{c^{n+1}}(\mathcal{B})$.

Let $h$ be the Coxeter number of $D$. Since $(c^{n+1})^h = w_0^{2(n+1)}$, our result implies that $\text{DT}_{D \boxtimes A_n}^{2(h+n+1)} = \text{Id}$. 
Thank you!


