Orthogonal Representations: from groups, through Hopf algebras, to tensor categories

Susan Montgomery

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Let G be a finite group and V be a finite-dimensional irreducible representation of G over $\mathbb{C}$. V is called **orthogonal** if it admits a non-degenerate G-invariant symmetric bilinear form.

Equivalently, V is defined over $\mathbb{R}$.

G is called **totally orthogonal** if all irreducible representations V of G are orthogonal.

Example: G is any finite real reflection group.

Definition: Let V be an irrep of G with character χ. The **n**th **Frobenius-Schur indicator** of V is defined as

$$\nu_n(V) := \frac{1}{|G|}\sum_{g\in G}\chi(g^n) = \chi(\frac{1}{|G|}\sum_{g\in G} g^n).$$

Frobenius-Schur Theorem (1906) $\nu_2(V) \in \{0, 1, -1\}$.
$\nu_2(V) \neq 0 \iff V^* \cong V$ and in that case
$\nu_2(V) = +1$ iff V admits a G-invariant symmetric non-degenerate bilinear form, and
$\nu_2(V) = -1$ iff the form is skew-symmetric.
$\nu_2(V) = 0 \iff V$ does not admit a G-invariant non-degenerate bilinear form

Isaacs (1960): $\nu_n(V) = \dfrac{1}{|G|}\displaystyle\sum_{g\in G} \chi(g^n) \in \mathbb{Z}$, all n.

FS, I: For all n, $\displaystyle\sum_V \nu_n(V)\text{dim}(V) = |\{x \in G \mid x^n = 1\}|$.

Scharf (1991): For $G = S_m$, all $\nu_n(V) \geq 0$, for all irreps V and all n.

Example: Consider the dihedral group D_4 and the quaternion group Q_8. Both have a unique 2-dim simple module, say V_1 for D_4 and V_2 for Q_8, and their group algebras have isomorphic Grothendieck rings. However $\nu_2(V_1) = +1$ and $\nu_2(V_2) = -1$. What is going on?

We will consider $\mathcal{C} = Rep(G)$ under $\otimes$; it is a tensor category. Among other properties, $\mathcal{C}$ has duals, and in fact $V^{**} \cong V$ for $V \in \mathcal{C}$. However one may check that for the two groups above, the two isomorphisms $V_1^{**} \cong V_1$ and $V_2^{**} \cong V_2$ are different.

Hopf algebras:

Let $H = \{H, m, u, \Delta, \varepsilon, S\}$ be a semisimple Hopf algebra over $\mathbb{C}$. H acts on tensor products of modules via Δ. That is, if $\Delta(h) = \sum h_1 \otimes h_2 \in H \otimes H$, then $h \cdot (v \otimes w) = \sum h_1 \cdot v \otimes h_2 \cdot w$.

Writing $\Delta^{n-1}(h) = \sum h_1 \otimes h_2 \otimes \cdots \otimes h_n$,
we define $h^{[n]} := m \circ \Delta^{n-1}(h) = \sum h_1 h_2 \cdots h_n$.
For $H = kG$ and $g \in G$, $\Delta(g) = g \otimes g$ and so $g^{[n]} = g^n$.

$\Lambda \in H$ is an **integral** if $h\Lambda = \varepsilon(h)\Lambda$ for all $h \in H$. When H is semisimple, we may choose Λ so that $\varepsilon(\Lambda) = 1$

Example: $H = kG$. Then $\Lambda = \dfrac{1}{|G|} \sum_{g \in G} g$,

and $\Lambda^{[m]} = \dfrac{1}{|G|} \sum_{g \in G} g^m$.

Definition: Let V be an irreducible representation of H with character χ. The **n**th **Frobenius-Schur indicator** of V is $\nu_n(V) := \chi_V(\Lambda^{[n]})$.

Theorem: (Linchenko-M 2000) H a semisimple Hopf algebra with integral Λ and irrep V. Then
(1) $\nu_2(V) \neq 0 \iff V^* \cong V$ and in that case,
$\nu_2(V) = +1$ iff V admits an H-invariant symmetric non-degenerate bilinear form,
$\nu_2(V) = -1$ iff the form is skew-symmetric, and
$\nu_2(V) = 0 \iff V$ does not admit any H-invariant non-degenerate bilinear form.
(2) $\sum_V \nu_2(V) \dim(V) = Tr(S)$.

Theorem: (Kashina-Sommerhäuser-Zhu 06) Consider the action on $V^{\otimes n}$ of the cyclic permutation α, given by $$v_1 \otimes \cdots \otimes v_n \mapsto v_n \otimes v_1 \otimes \cdots \otimes v_{n-1}.$$
Then $(V^{\otimes n})^H$ is stable under the action of α, and

$$\nu_n(V) := trace(\alpha|_{(V^{\otimes n})^H}).$$

Thus $\nu_n(V) \in \mathcal{O}_n$, the ring of n^{th} cyclotomic integers. Moreover $\sum_V \nu_n(V)\mathrm{dim}(V) = Tr(S \circ P_{n-1})$.

Example: (KSZ) $\mathbb{Z}_9$ acts on A_4 (and so on $\mathbb{C}^{A_4}$) by conjugation by a fixed 3-cycle. Then $H = \mathbb{C}^{A_4}\#\mathbb{C}\mathbb{Z}_9$ has an irrep V so $\nu_3(V) = 1 + \zeta_3 \notin \mathbb{Z}$.

Applications

1. **Exponents**: For a Hopf algebra H, the **exponent** Exp(H) of H is the smallest positive integer m such that $x^{[m]} = \varepsilon(x)1$, for all $x \in H$.

Question: For H semisimple, does Exp(H) divide dim$H$?

True if H commutative or cocommutative (60's), $D(G)$ (K 97).

Theorem: (1) (Etingof-Gelaki 99) Exp(H)divides (dim$H$)^3.
(2) (KSZ 06) If a prime p divides dim(H), then p divides Exp(H)

(2) is a version of Cauchy's theorem. Their proof uses indicators

2. Classification: Dim 8 quasi-Hopf algebras over $\mathbb{C}$

(Masuoka) There are exactly eight semisimple dim 8 Hopf algebras: five group algebras, $\mathbb{C}(D_4)^*$, $\mathbb{C}(Q_8)^*$, and the Kac-Palyutkin algebra K_8.

(Tambara-Yamagami 98) There are exactly four fusion categories $Rep(H)$ which can arise from a non-commutative quasi-Hopf algebra H of dim 8.
Three of them are $\mathbb{C}D_8$, $\mathbb{C}Q_8$, K_8. What is the fourth?

For these categories $\mathcal{C} = Rep(H)$, $Irr(\mathcal{C}) = G \cup \{\rho\}$, where G is finite abelian, $gh = hg$ for all $g, h \in G$, $g\rho = \rho g = \rho$, and $\rho^2 = \sum_{g \in G} g$. Such categories are called **Tambara-Yamagami** categories.

(NSch 05) Construct a quasi-Hopf algebra "twist" $(K_8)^u$. The 2-dim rep V has indicators $\{\nu_2(V), \nu_4(V)\}$ which do not match any of the others.

The Drinfel'd double of a finite group G:

$$D(G) = k^G \bowtie kG$$

As an algebra, $D(G)$ is the semi-direct product $k^G \# kG$, where k^G is the function algebra and the action of G on k^G is induced from the conjugation action of G on itself. As a coalgebra, $D(G)$ is the tensor product of the coalgebras k^G and kG.

Representations of $D(G)$ (DPR, Ma 90):

Fix an element u in each conjugacy class of G and let $C(u)$ be the centralizer of u in G. Let W be an irreducible $C(u)$-module and define $V := \mathbb{C}G \otimes_{\mathbb{C}C(u)} W$. With a suitable action of $\mathbb{C}^G$ on V, V is an irreducible $D(G)$-module. All irreducible modules arise in this way.

Recall Scharf proved that for $G = S_m$, all $\nu_n(V) \geq 0$. Is this true for $D(G)$?

(1) (Guralnick-M 09) $D(G)$ is totally orthogonal for any finite real reflection group G; (K-Mason-M 02) $G = S_m$.

(2) (Keilberg 10) For $H = D(D_m)$, all $\nu_n(V) \in \mathbb{Z}_{\geq 0}$.

(3) (Courter 12) For $H = D(S_m)$, $m \leq 12$, all $\nu_n(V) \in \mathbb{Z}_{\geq 0}$. (Schauenburg 15) True for $m \leq 23$

Question: For $H = D(G)$, when are all values of $\nu_n(V) \in \mathbb{Z}$?

Definition (KSZ): Define
$G_n(u,g) := \{a \in G \mid (au^{-1})^n = a^n = g\}$,
where u is in a fixed conjugacy class, W is an irrep of $C = C(u)$, and V is the induced module for $D(G)$. Let η be the character of W and χ_η be the character of V.

Theorem (Iovanov-Mason-M 14): All indicators for $D(G)$ are in $\mathbb{Z} \iff$ for all commuting pairs $u, g \in G$ and all n such that $gcd(n, |G|) = 1$,

$$|G_n(u,g)| = |G_n(u,g^n)|.$$

Examples: $G = PSL_2(q)$, A_m, S_m, M_{11}, M_{12}, or if G is a regular p-group.

False for the Harada-Norton simple group and for the Monster.

Tensor categories

We assume here that $\mathcal{C}$ is a spherical rigid fusion category; that is, $\mathcal{C}$ is a semisimple category with a finite number of simples, and it has duals.

Spherical means that the left and right traces coincide. For example, consider the category $\mathcal{C} = Vec$ of finite-dim vector spaces over $\mathbb{C}$. The spherical structure $j : V \to V^{**}$ is the natural isomorphism of vector spaces, $ev : V^* \otimes V \to \mathbb{C}$ is the usual evaluation map and $coev : \mathbb{C} \to V \otimes V^*$ is the dual basis map. For any $f : V \to V$, the categorical trace of f is the composition map

$$\mathbb{C} \to V \otimes V^* \to V \otimes V^* \to V^{**} \otimes V \to \mathbb{C}$$

where the first map is $coev$, the second $f \otimes id$, the third $j \otimes id$, and the last ev. This trace is identical to the ordinary trace of f.

In general a fusion category is determined up to equivalence by its fusion rules and by the "6j symbols". These symbols are all the isomorphisms in the tensor category axioms, Thus the actual isomorphisms

$$(V \otimes W) \otimes X \cong V \otimes (W \otimes X)$$

for $V, W, X \in \mathcal{C}$, are important.

A property is a **gauge invariant** if it is invariant under equivalence of categories. **Ng - Schauenburg 07** show that FS-indicators can be extended to these categories using traces, extending KSZ's definition.
They also showed that indicators are gauge invariants (done earlier by Mason and Ng for quasi-Hopf algebras.)

Recall a fusion category $\mathcal{C}$ is **TY** if $Irr(\mathcal{C}) = G \cup \{\rho\}$, where G is finite abelian, $gh = hg$ for all $g, h \in G$, $g\rho = \rho g = \rho$, and $\rho^2 = \sum_{g \in G} g$.

A **near group** has the same relations as above except that $\rho^2 = \sum_{g \in G} g + m\rho$, for $m = |G| - 1$ or $k|G|$.

Definition (Tucker): A fusion category is **FS-indicator rigid** if it is determined by its fusion rules and all of its indicators.

Theorem (Basak-Johnson 2015): TY-categories are FS-indicator rigid.

Theorem (Tucker 2015) If $\mathcal{C}$ is a near group with $m = |G| - 1$, then $\mathcal{C}$ is FS-indicator rigid. The same is true for $m = |G|$ when the center of $\mathcal{C}$ is known.

Izumi-Tucker The non-commutative near-group fusion rings also have FS-indicator rigidity.

False for more general categories, such as Haagurup-Izumi categories