4/28/19 (1
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Actions of Hopf algebras on matrices

Let k be a field, usually $\mathbb{C}$

$H = (H, m, u, \Delta, \varepsilon, S)$ is a Hopf algebra/k

and A is an algebra/k

Def: H acts on A if

① A is a (unital) left H-module

② for $h \in H$, $a, b \in A$, and $\Delta h = \sum h_1 \otimes h_2$

$$h\cdot(ab) = m\circ(\Delta h\cdot(a\otimes b))$$
$$= \sum (h_1\cdot a)(h_2\cdot b)$$

Ex's ① $H = kG$, $\varepsilon(g)=1$

Then for $g \in G$, $\Delta g = g\otimes g$, $g\cdot(ab) = (g\cdot a)(g\cdot b)$

Thus g acts as an automorphism

② $H = U(L)$, L Lie alg.

$x \in L$ is primitive if $\Delta x = x\otimes 1 + 1\otimes x$

$x\cdot(ab) = (x\cdot a)\cdot b + a(x\cdot b)$, a derivation

③ $x \in H$ is skew-primitive if for some $g \in G(H)$, $\Delta x = x\otimes 1 + g\otimes x$.

Then $x\cdot(ab) = (x\cdot a)b + (g\cdot a)(x\cdot b)$

a skew derivation

Ex: Taft algebras. Let $\omega \in k$ be a primitive n^{th} root of 1. Then

$$H_{n^2}(k) = {}_k\langle g, x \mid g^n = 1, x^n = 0, xg = \omega g x, \; g \in G(H), \Delta x = x\otimes 1 + g\otimes x, \; \varepsilon(x) = 0\rangle$$

<u>Def</u>: H is <u>pointed</u> if all simple (minimal) subcoalgebras are 1-dim.
When H is finite dim, this means that in H^*, $H^*/\mathrm{rad}(H^*)$ is <u>basic</u>: a sum of fields.

<u>ex</u>: Lusztig's small quantum group.

$$u_q(sl_2) = k\langle E, F, K \mid E^n = 0 = F^n, K^n = 1, KE = q^2 EK, KF = q^{-2} FK, \text{ and } EF - FE = \frac{K - K^{-1}}{q - q^{-1}} \rangle$$

$\dim_k(u_q(sl_2)) = n^3$ when $o(q) = 2n$.
E, F are skew-primitive.

Note $u_q(sl_2)$ contains 2 copies of Taft alg

<u>Conjecture</u> (Andruskiewitsch, Schneider). Any fin. dim pointed Hopf algebra is generated as an algebra by group-likes and skew-primitives.

<u>Angiono</u>: True if G finite abelian.

[AS]: showed true for most cases of ↑, but needed very difficult classification.

<u>Ex</u>: Drinfeld double of Taft algebra.

$$D(H) = {}_k\langle x, g, X, G \mid gG = Gg, Xg = \omega^{-1} gX, xG = \omega^{-1} Gx, xX - Xx = G - g \rangle$$

There is a surjection $D(H) \twoheadrightarrow u_q(sl_2)$ as follows:

set $\omega = q^{-2}$ and let:

$G \rtimes 1 \to K$, $\varepsilon \rtimes g \to K^{-1}$, $\varepsilon \rtimes x \to F$

$X \rtimes 1 \to -(q - q^{-1}) E$.

Notice that we must have $G \rtimes g \longrightarrow 1$.

<u>M-Smith</u> (1991). Determined all actions of $U_q(sl_2)$ on polynomials $k[z]$ in generic case

analog of actions of sl_2 (and $U(sl_2)$).

<u>M-Schneider</u> (2001). Determined all actions of

$H = T_{n^2}(\omega)$ on $A = k[u \mid u^n = \beta]$, some $\beta \in k$.

<u>Question</u>: what if A has two generators?

$$A = k\langle u, v \mid u^n = \alpha, v^n = \beta, vu = \gamma uv\rangle$$

for suitable k, γ.

See that $A \cong M_n(k)$,

Recall that A is generated as an algebra by two matrices. A nice example of generators are the <u>Sylvester matrices</u>

$$C = \begin{bmatrix} 1 & & & 0 \\ & \varepsilon & & \\ & & \varepsilon^2 & \\ 0 & & & \ddots \ \varepsilon^{n-1} \end{bmatrix}, \quad S = \begin{bmatrix} 0 & 1 & & 0 \\ & 0 & 1 & \\ & & \ddots & 1 \\ 1 & & & 0 \end{bmatrix}$$

where $\varepsilon^n = 1$. See that $SC = \varepsilon CS$.

Thus a nice basis of $M_n(k)$ is $\{X_{kl} = C^k S^l\}$.

<u>Joint work with Yuri Bahturin</u>

Action of g on A replaced by a <u>grading</u> of A by $\langle g\rangle$, as $kG \cong (kG)^*$ *

<u>Theorem</u> (2001) (Bahturin, Sehgal, Zaicev)

Let G be a finite abelian group, and assume that $M_n(F)$ is graded by G.

Then $M_n(F) \cong \underbrace{M_k(F)}_{A} \otimes \underbrace{M_t(F)}_{B}$

where

① B has an <u>elementary grading</u>:

That is, write $G = \{g_1, \dots, g_t\}$.

Then $B_g = \mathrm{span}_k \langle E_{ij} \mid g_i^{-1} g_j = g \rangle$

for <u>example</u>, $G = \mathbb{Z}_t$ gives a grading on B. Say $\mathbb{Z}_t = \langle a \rangle = \{1, a, \dots, a^{t-1}\}$

Then $B_a = \begin{bmatrix} 0 & 1 & & 0 \\ & & \ddots & \\ & & & 1 \\ 1 & & & 0 \end{bmatrix}$, $B_{a^2} = \begin{bmatrix} 0 & 0 & 1 & \\ & & \ddots & 1 \\ & & & 0 \\ 1 & 0 & & 0 \end{bmatrix}$ etc.

② $A = M_k(F)$ has a <u>fine grading</u>.

This means $\dim A_g = 1$, all $g \in G$.

The <u>support</u> of G on A is a subgroup of order k^2.

See book by Elduque & Kochetov

<u>ex</u>: $A = M_2(F)$, $G = \mathbb{Z}_2 \times \mathbb{Z}_2$

say $G = \{e, a, b, c \mid a^2 = b^2 = c^2 = e,$ and G abelian$\}$

$A_a = k\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, $A_b = k\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $A_c = k\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

$A_e = k\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Note $c^2 = -e$, but OK for spaces: $A_c^2 = A_e$.

For a fine grading to exist, need that G is "square"; that is,

$$G = \mathbb{Z}_{n_1}^2 \times \mathbb{Z}_{n_2}^2 \times \cdots \times \mathbb{Z}_{n_\ell}^2$$

<u>Another simplification</u>:

<u>Theorem (Masuoka)</u>: If H acts on $M_n(k)$, Then the action is <u>inner</u>, That is, There is a (linear) function $u: H \to M_n(k)$ such that for $a \in M_n(k)$,

$$h.a = \sum u(h_1)\, a\, u(Sh_2)$$

(recall the antipode S of H is a kind of inverse).

For our case,

$g \in G(H) \Rightarrow g.a = u(g)\, a\, u(g)^{-1}$

x skew-primitive $\Rightarrow$

$$x.a = u(x)a - u(g)\, a\, u(g)^{-1}.\, u(x)$$

<u>Lemma</u>: replace $u(x)$ by $u'(x) = u(x) - \lambda u(g)$

Thus <u>may assume</u>:

$u(g) = 1_A$, $u(x)^n = \alpha . 1_A$, $u(x)u(g) = \omega\, u(g) u(x)$.

<u>Lemma</u>: If the action of g is not faithful, then $x.A = 0$.

use the special basis $\{X_{k\ell}\}$.

<u>Theorem</u> (Bahturin – M). Determine all actions of $T_9(\omega)$ on $M_3(\mathbb{C})$.

Also actions of $T_4(\omega)$ & $u_q(sl_2)$ on $M_2(\mathbb{C})$

$\uparrow$ Gordienko

ex: In previous work of M-Schneider on $A = k[u \mid u^n = \alpha]$, showed that action of $D(T_{n^2}(\omega))$ determines action of $u_q(sl_2)$; all $u_q(sl_2)$ actions come from $D(T_{n^2}(\omega))$.

new ex: This is false for 2-generator A.

There is an action of $D(T_4(\omega))$ on $M_2(\mathbb{C})$ via:

$$u(g) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad u(G) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

$$u(x) = -\tfrac{1}{2}\, u(X) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

Note $gG^{-1} \neq 1$ so can't have a $u_q(sl_2)$ action.

ex: Actions of $T_n(\mathbb{C})$ on $M_3(\mathbb{C})$.

write $u(g) = Q$ and $u(x) = P$.

3 cases for Q:

① $Q_1 = (I, \omega I_2)$, $P_1 = E_{13}$

② $Q_2 = (I, \omega^2 I_2)$, $P_2 = E_{31}$

③ $Q_3 = (1, \omega, \omega^2)$, $P_{3,i}$, $i = 1, \dots, 6$,

see $P_{3i} = E_{12}$, or E_{23}, or E_{21}, or $E_{12} + E_{23}$, $E_{12} + E$, $E_{23} + E_3$

or an infinite family of P's:

$P_{3,\gamma} = E_{12} + E_{23} + \gamma E_{31}$, any $\gamma \in F^*$

These actions are all inequivalent.