Representations of quantum groups at \( p^r \)th root of 1 over \( p \)-adic fields

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I. Various representation theories of algebraic groups

The groups

- Let $G$ be a reductive algebraic group defined over $\mathbb{F}_q$ and $k = \overline{\mathbb{F}}_q$.

Example: $GL_n$ is defined over $\mathbb{Z}$. For any commutative ring $A$, $GL_n(A)$ is the group of all invertible matrices in $A$.

Ring homomorphism $f : A \to B$ gives a group homomorphism

$$GL_n(f) : GL_n(A) \to GL_n(B).$$
There are many groups associated to $G$ by taking rational points over various fields:

- Finite groups $G(q^r) = G(F_{q^r})$
- Infinite groups $G = G(k)$ for any field extension $k \supseteq F_q$
- The groups $G(F_q[t]/t^n)$ and the limit $G(F_q[[t]]) \subseteq G(F_q((t))$
- The groups $G(\overline{F}_q[t]/t^n)$ and the limit $G(\overline{F}_q[[t]]) \subseteq G(\overline{F}_q((t))$
- $p$-adic groups $G(Q_p)$

- Profinite groups and proalgebraic groups Consider smooth representations.

- Representation theory of $G(q^r)$ over a field $K$: The classical question: for characteristics of $K$ being the same as that of $F_q$ or different.
• Rational representation theory of $G$ (representations over $k$), one of the main topics.

• Representations of the infinite groups $G = G(k)$ as an abstract group over a field $K$.

• Representations of the Lie algebra $g = \text{Lie}(G)$ (over the defining field $k$), both restricted representations and other representations.

Example: For $G = \text{GL}_n$, $g = \mathfrak{gl}_n(k) = \text{End}_k(k^n)$. The restricted structure is the map $x \mapsto x^p \in \text{End}_k(k^n)$.

• Representations of the Frobenius kernels $G_r$ and their thickenings.
Example: For $G = GL_n$, $G_r(A) = \ker(Fr : G(A) \to G(A))$ with $Fr((a_{ij}) = (a_{ij}^q)$.

• Representations of the hyperalgebra (or distribution algebra) $D(G) = \text{Dist}(G)$ and its finite dimensional sub-algebras $D_r(G) = \text{Dist}(G_r)$.

Example: For $G = G_a$,

$$\text{Dist}(G) = k\text{-span}\{x^{(n)} \mid n \in \mathbb{N}\}/ \sim$$

$$x^{(n)}x^{(m)} = \binom{n + m}{n} x^{(n+m)}$$

"think of" $x^{(n)} = x^n/n!$

$\text{Dist}(G_r) = k\text{-span}\{x^{(n)} \mid n < q^r\}$
Example: For $G = G_m$,

$$\text{Dist}(G) = k\text{-span}\{\delta(n) \mid n \in \mathbb{N}\}$$

$$\delta(n)\delta(m) = \sum_{i\geq 0} \binom{n + m - i}{n - i, m - i, i} \delta(n+m-i)$$

"think of" $\delta(n) = \binom{\delta_1}{n}$

$$\text{Dist}(G_r) = k\text{-span}\{\delta(n) \mid n < q^r\}.$$
• Relations among these representation theories are complicated. Some of them have quantum analog and others, not known yet.

• Representations of $G(q^r)$ over $k$ and that of $D_r(G)$ and $G_r$, and rational representations are well studied. Irreducibles, projectives, cohomology theories etc.

• Representations of $G(q^r)$ over $\mathbb{C}$, or $\bar{\mathbb{Q}}_l$ ($l \neq p$) for all $r$. Character theory controls everything: How to compute the characters? directly compute, one group at a time. Deligne-Lusztig characters, and Lusztig’s character sheaf theory: certain perverse sheaves on the algebraic variety $G(k)$ (constructible $l$-adic sheaves with values in $\bar{\mathbb{Q}}_l$).
• Representations of $G'(q^r)$ and over $K = \bar{\mathbb{K}}$ with $\text{ch}(K) \neq \text{ch} (\mathbb{F}_q)$, there are also geometric approach by considering the constructible sheaves with coefficient in $K$ by Juteau and many others using Langland dual group.

**Theorem 1** (Borel-Tits-1973). Let $G$ and $G'$ be two simple algebraic groups over two different fields $k$ and $k'$ respectively. If there is an abstract group homomorphism $\alpha : G(k) \to G'(k')$ such that $\alpha([G, G])$ is dense in $G'(k')$, then $\alpha$ “almost” rational algebraic group homomorphism. In particular there is field homomorphism $k \to k'$ and $\text{char}(k) = \text{char}(k')$.

Essentially if $E$ and $k$ have different characteristic, the infinite group $G(k)$ does not have finite dimensional non-trivial representations.
Example 1. Let $G = \mathbb{G}_m = GL_1$ be the multiplicative group scheme. $G(k) = k^\times$.

$W_p(k)$ — the ring of Witt vectors of the field $k$.

$K$ — the field of fractions of $W_p(k)$.

Then the commutative group $\mathbb{G}_m(k)$ has plenty one dimensional representations. For example, the Teichmüller representative $\tau : k^\times \to W_p(k)^\times \subset GL_1(K)$ is a group character. The Galois groups $\text{Gal}(k)$ acts on the set of all characters.

Remark: $W_p(\mathbb{F}_p) = \mathbb{Z}_p$, the $p$-adic integers, $K = \mathbb{Q}_p$.

More general

$$\text{Det} : GL_n(k) \to k^\times \xrightarrow{\tau} W_p(k)^\times \subset GL_1(K).$$
Example 2. \( G = \mathbb{G}_a, \ \mathbb{G}_a(k) = (k, +) \). Fix any \( \rho \)th root \( \xi \in K \) of 1, \( \psi : \mathbb{Z}/p\mathbb{Z} \to \mu_p \subseteq K^\times \) by \( \psi(n) = \xi^n \). \( k \) is a \( \mathbb{F}_p \) vector space and choose a basis, one has non-countablely many irreducible representations if \( \text{Ch}(K) \neq p \) and one single irreducible representation if \( \text{Ch}(K) = p \).

Remark 1. \( G(k) = \bigcup_{r \geq 1} G(q^r) \) is a union of finite groups.

Reductive groups are built up from \( \mathbb{G}_m \)'s and \( \mathbb{G}_a \)'s through the root systems.

There are subgroups \( G \supset B = T \ltimes U \) and \( W = N_G(T)/T \) all defined over \( \mathbb{F}_q \) and they have corresponding subgroups of rational points.
The representations of the infinite group $G(k)$ were considered by Nanhua Xi in 2011 using the fact that $G(k)$ is a directed union of finite groups of Lie type.

The standard constructions of induced representations and Harish-Chandra induced representations have interesting decompositions (with finite length). But induced modules are no longer semisimple (even over $\mathbb{C}$) and the Hecke algebras are trivial.

**Example** The induced module $KG(\overline{F}_p) \otimes_{KB(\overline{F}_p)} K$ has only finitely many composition factors indexed by subsets of simple roots and each appears exactly once in all characteristics. But $\text{End}(KG(\overline{F}_p) \otimes_{KB(\overline{F}_p)} K) = K$. The Hecke algebra is trivial even for $K = \mathbb{C}$. 
When $K = k$, then both finite dimensional representations (rational representations) and non-rational representations (infinite dimensional representations) all appear.

**Remark 2.** $D(G) = \bigcup_{r \geq 1} D_r(G)$ is also a union of finite dimensional Hopf subalgebras.

The goal is to relate representations of $D(G)$ and that $G(k)$ over $k$, in terms of Harish-Chandra inductions. The best analog is the category $\mathcal{O}$ of the Hyperalgebra $D(G)$. 
II. Irreducible characters in category $\mathcal{O}$

Let $U = \text{Dist}(G)$ Then $U = U^- \otimes_k U^0 \otimes_k U^+$, as $k$-vector space.

The commutative and cocommutative Hopf $k$-algebra $U^0 = \bigotimes \text{Dist}(\mathbb{G}_m)$ (not finitely generated) defines an abelian group scheme $X = \text{Spec}(U^0)$ with group operation written additively. Let $X(k)$ denote the $k$-rational points of $X$.

Kostant $\mathbb{Z}$-form defines a $\mathbb{Z}$ structure on $X$ and $X(K) = (\mathfrak{h}_\mathbb{Z} \otimes_\mathbb{Z} K)^*$ if char($K$) = 0 and $X(k) = X(W_p(k)) \supseteq X(\mathbb{Z}_p)$. 
$X(k) = X(W_p(k))$ is a free $W_p(k)$-module with a basis $\{\omega_i\}$ (the fundamental weights).

If $Q = \mathbb{Z}\Phi$ is the root lattice, then there is a paring $Q \times X(k) \to W_p(k)$ with $(\alpha, \lambda) = \langle \alpha^\vee, \lambda \rangle$.

$$0 \to p^r X(k) \to X(k) \to X_r \to 0$$

- Verma modules $M(\lambda) = U \otimes_{U \geq 0} k\lambda$ with $\lambda \in X(k)$.
- $M(\lambda)$ has unique irreducible quotient $L(\lambda)$.

Inductive limit property:
• \( M(\lambda) = \bigcup_{r=1}^{\infty} \text{Dist}(G_r)v_\lambda^+ \).

• \( L(\lambda) = \bigcup_{r=1}^{\infty} \text{Dist}(G_r)v_\lambda^+ \).

• Each module \( M \) in the category \( \mathcal{O} \) defines function \( \text{ch}_M : X(k) \to \mathbb{N} \), written as formal series:

\[
\text{ch}_M = \sum_{\lambda \in X(k)} \dim(M_\lambda)e^\lambda.
\]

• One has to replace group algebra \( \mathbb{Z}[X(k)] \) by function algebra with convex conical supports on \( X(k) \) in order for convolution product to make sense.
• Frobenius morphism \( Fr : G \to G \) over \( \mathbb{F}_q \) defines a map \( X(k) \to X(k) \) \((\lambda \mapsto \lambda^{(1)} = q\lambda)\). Similarly \( \lambda^{(r)} = q^r \lambda \) Frobenius twisted representation.

**Theorem 2** (Haboush 1980). For each \( \lambda = \sum_{r=0}^{\infty} p^r \lambda^r \in X(k), \)

\[
L(\lambda) = L(\lambda^0) \otimes L(\lambda^1)^{(1)} \otimes L(\lambda^2)^{(2)} \otimes \ldots
\]

Infinite tensor product should be understood as direct limit.

**Goal:** compute the character \( \text{ch}_{L(\lambda)} \) in terms of the function \( \text{ch}_{M(\mu)} \).
Haboush theorem implies

\[ \text{ch}_\lambda = \prod_{r=1}^{\infty} (\text{ch}_{L(\lambda^r)})(r). \]

The infinite product makes sense in the function spaces.

**Example 3.** Let \( \lambda = -\rho \in X(\mathbb{Z}) \subseteq X(\mathbb{Z}_p) = X(k) \). Then \( L(-\rho) = M(-\rho) = L((q-1)\rho) \otimes L((q-1)\rho)^{(1)} \otimes L((q-1)\rho)^{(r)} \otimes \ldots \) using the fact \(-1 = \sum_{r=0}^{\infty} (q-1)q^r\).
III. Generic quantum groups over a $p$-adic field—Nonintegral weights

- Let $\mathbb{Q}_p' = \mathbb{Q}_p[\xi]$ where $\xi$ is a $p^r$-th root of 1.

- $\mathbb{Q}_p'$ is a discrete valuation field and let $\mathbb{A}$ be the ring of integers in $\mathbb{Q}_p'$ over $\mathbb{Z}_p$. Then $\mathbb{A}$ is a complete discrete valuation ring with maximal ideal $p\mathbb{A}$ generated by $p$.

- Each $\lambda \in \mathbb{Z}_p$ defines a $\mathbb{Q}_p'$ algebra homomorphism $\mathbb{Q}_p'[K, K^{-1}] \to \mathbb{Q}_p'$ by sending $K \to \xi^\lambda$.

- $\xi^\lambda \in \mathbb{A}$. In fact $\xi \in \mathbb{Q}_p'$ is a $p^r$th-root of 1 implies $z = \xi - 1 \in p\mathbb{A}$ and

$$
(1 + z)^\lambda = \sum_{n=0}^{\infty} \binom{\lambda}{n} z^n \text{ converges in } \mathbb{Q}_p', \forall \lambda \in \mathbb{A}.
$$
• For an indeterminate \( v \), set \( z = v - 1 \in Z[v, v^{-1}] \). 

\[
v^{-1} = \sum_{n=0}^{\infty} (-1)^n z^n \in A[[z]] \text{ implies } Z[v, v^{-1}] \subseteq A[[z]]
\]
and \( Q(v) \subseteq Q_p((z)) \). For any \( \lambda \in Z_p[[z]] \)

\[
v^{\lambda} = \sum_{n=0}^{\infty} \binom{\lambda}{n} z^n
\]
is convergent in \( Z_p[[z]] \) by noting that \( \binom{\lambda}{n} \in Z_p[[z]] \).

• Let \( U_C(v) \) (generic case) be the quantum enveloping algebra of \( g_C \) over the field \( C(v) \). Let \( U_{Z[v,v^{-1}]} \) be the \( Z[v, v^{-1}] \)-form in \( U_C(v) \) constructed by Lusztig using divided powers.

• Set \( U_{Q'_p} = U_{Z[v,v^{-1}]} \otimes_{Z[v,v^{-1}]} Q'_p \) and \( U_{Q'_p((z))} \) and
\( U_A((z)) \) etc. They all have compatible triangular decompositions.

- The subring \( U_0^0 \mathbb{Z}[v,v^{-1}] \) is a commutative and cocommutative Hopf algebra over \( \mathbb{Z}[v,v^{-1}] \)

- Each \( \lambda = (\lambda_i) \in \mathbb{Q}_p((z))^I \) defines a \( \mathbb{Q}_p((z)) \)-algebra homomorphism

\[
\lambda : U_{\mathbb{Q}_p((z))}^0 \to \mathbb{Q}_p((z)) \quad K_i \mapsto v_i^{\lambda_i}.
\]

Then \( \lambda(U_A[[z]]) \subseteq A[[z]] \) if \( \lambda \in A[[z]]^I \) and \( \lambda(U_{\mathbb{Z}_p[[z]]}) \subseteq \mathbb{Z}_p[[z]] \) if \( \lambda \in \mathbb{Z}_p[[z]]^I \).
For $\lambda \in \mathcal{Q}'((z))^I$, the quantum Verma module for the algebra $U_{\mathcal{Q}_p'}((z))$ is

$$M_{\mathcal{Q}_p'}((z))((\lambda)) = U_{\mathcal{Q}_p'}((z)) \otimes_{\mathcal{Q}'((zz))} \mathcal{Q}'((z))^\lambda$$

with irreducible quotient $L_{\mathcal{Q}_p'}((z))((\lambda))$. The characters are similarly defined as functions $\mathcal{Q}_p'((z))^I \to \mathbb{Z}$.

Standard argument implies $L_{\mathcal{Q}_p'}((z))((\lambda)) = M_{\mathcal{Q}_p'}((z))((\lambda))$ unless $\langle \tilde{\alpha}, \lambda + \rho \rangle \in \mathbb{Z}_{\geq 0} \subseteq \mathcal{Q}_p'((z))$. In general we have

$$\text{ch } L_{\mathcal{Q}_p'}((z))((\lambda)) = \text{ch } \Delta_{\mathcal{Q}_p'}((z))((\lambda)).$$

Here $\Delta_{\mathcal{Q}_p'}((z))((\lambda))$ is the irreducible $g_{\mathcal{Q}_p'}((z))$-module.

The characters $\text{ch } \Delta_{\mathcal{Q}_p'}((z))((\lambda))$ can be determined by an argument similar that in the category $\mathcal{O}$ for $g_{\mathbb{C}}$ as
outlined in Humphreys’ book by replacing the field $\mathbb{C}$ with $\mathbb{Q}_p((z))$.

- The generalized Kazhdan-Lusztig conjecture for non-regular blocks $(O_{\mathbb{Q}_p((z))})_\lambda$ gives the following decomposition of characters

$$
\text{ch} L_{\mathbb{Q}_p((z))}(\lambda) = \sum_\mu p_{\mu,\lambda}^0 \text{ch} M_{\mathbb{Q}_p((z))}(\mu) \quad (1)
$$
IV. Quantum groups at $p^r$th roots of unit over a $p$-adic field

- Let $\xi$ be a $p^r$th root of 1.
- The map $Q'_p[[z]] \rightarrow Q'_p (z \mapsto \xi - 1)$ induces $A[[z]] \rightarrow A$. Define

$$U_{Q'_p} = U_{Z[v,v^{-1}]} \otimes_{Z[v,v^{-1}]} Q'_p = U_{A[[z]]} \otimes_{A[[z]]} Q'_p$$

with $A$-form $U_{A_p} = U_{A[[z]]} \otimes_{A[[z]]} A$ with tensor product decomposition

$$U_A = U_A^- \otimes_A U_A^0 \otimes_A U_A^+.$$
• Let $\mathcal{O}_{Q_p}$ be the category $\mathcal{O}$ construction by Andersen and Mazorchuk for the quantum group $U_{Q_p}$.

• The Verma module $M_{Q_p}(\lambda)$ and irreducible quotient $L_{Q_p}(\lambda)$ in $\mathcal{O}_{Q_p}$ with $\lambda \in X(\mathbb{Z}_p) \subseteq X(\mathbb{Q}_p)$.

• For $\lambda \in X(\mathbb{Z}_p)$, $L_{A_p[[z]]}(\lambda) = U_{A_p[[z]]} v_\lambda^+ \subseteq L_{Q_p((z))}(\lambda)$ is an $A[[z]]$-lattice.

• Define $V_{Q_p}(\lambda) = L_{A_p[[z]]}(\lambda) \otimes_{A_p[[z]]} Q_p'$ to be the Weyl module with the surjective maps $M_{Q_p}(\lambda) \rightarrow V_{Q_p}(\lambda) \rightarrow L_{Q_p}(\lambda)$. 
Proposition 1 (Andersen-Mazorchuk). For any \( \lambda = \lambda' + p\lambda'' \in X(k) \) with \( \lambda' \in X_1 \),

\[
L_{Q'_p}(\lambda) = L_{Q'_p}(\lambda') \otimes (\Delta_{Q'_p}(\lambda''))^{(1)}.
\]

Taking \( A \)-lattices generated by highest weight vectors and then tensor with \( A \to k \), we get representations of \( \text{Dist}(G) \)

Proposition 2. For \( \lambda = \lambda' + p\lambda'' \in X(k) \),

\[
\overline{L_{A_p}(\lambda)} = \overline{L_{A_p}(\lambda')} \otimes \Delta(\lambda'')^{(1)}.
\]
V. Decomposition Multiplicities in Quantum Verma Modules

• For $\lambda \in X(\mathbb{Z}_p)$, define.

$$E_0^\lambda = \text{ch} \Delta(\lambda) = \text{ch} \Delta_{Q_p}^\lambda(\lambda).$$

Here $\Delta_{Q_p}^\lambda(\lambda)$ is the irreducible representation of the Lie algebra $g_{Q_p}$ with "A-integral" highest weight $\lambda$.

• For each $r \geq 0$, any $\lambda \in X(\mathbb{Z}_p)$ can be uniquely written as $\lambda' + p^r \lambda''$ with $\lambda \in X_r$. Define recursively

$$E_{\lambda}^{k+1} = \sum_{\mu \in X(k)} p_{\mu, \lambda''} E_{\lambda'}^k + (p)^{k} \mu.$$  \hfill (2)
Standard argument by Lusztig to get:

\[ E^k_\lambda = \sum_{\mu \in X(k)} d^q_{\mu, \lambda''} E^{k+1}_{\lambda'} + p^r \mu, \]

\[ E^k_\lambda = E^1_{\lambda^0} (E^1_{\lambda^1})^{(1)} \cdots (E^1_{\lambda^{k-1}})^{(k-1)} (E^0_{\lambda^k} \sum_{j \geq k} p^r (j-k) \lambda_j)^{(k)}. \]

- Define

\[ E^\infty_\lambda = E^1_{\lambda^0} (E^1_{\lambda^1})^{(1)} \cdots (E^1_{\lambda^{k-1}})^{(k-1)} (E^1_{\lambda^k})^{(k)} \cdots . \]  \hspace{1cm} (3)

- Recursively define \( F^k_\lambda \) as follows: \( F^0_\lambda = \text{ch } M(\lambda) \) and for \( k \geq 0 \)

\[ F^{k+1}_\lambda = \sum_{\mu \in X(k)} a^q_{\mu, \lambda''} F^k_{\lambda'} + (p)^k \mu. \]  \hspace{1cm} (4)
Lusztig’s argument implies

\[ F^k_\lambda = \sum_{\mu \in X(k)} d^q_{\mu, \lambda''} F^{k+1}_{\lambda' + p^r \mu}. \]

\[ F^k_\lambda = F^1_{\lambda_0} (F^1_{\lambda_1})^{(1)} \cdots (F^1_{\lambda_{k-1}})^{(k-1)} (F^0_0 \sum_{j \geq k} p^r (j-k) \lambda_j)^{(k)}. \]

- As before, the infinite product converges in \( F[X(k)] \).

Note that \( E^1_\lambda = F^1_\lambda = \text{ch} \ L_q(\lambda) \) for all \( \lambda \). We have \( E_\lambda^\infty = F_\lambda^\infty \). But for other \( k \), \( E^k_\lambda \) and \( F^k_\lambda \) are different.

**Proposition 3.** For any \( k \), both sets \( \{ E^k_\lambda \mid \lambda \in X(\mathbb{Z}_p) \} \) and \( \{ F^k_\lambda \mid \lambda \in X(\mathbb{Z}_p) \} \) are basis of \( F[X(\mathbb{Z}_p)] \).
We define the following decomposition of characters
\[ F^k_\lambda = \sum_{\mu \in X(\mathbb{Z}_p)} d^{(k)}_{\mu,\lambda} E^k_{\mu}; \]
\[ E^k_\lambda = \sum_{\mu \in X(\mathbb{Z}_p)} a^{(k)}_{\mu,\lambda} F^k_{\mu}. \]

- For each fixed \( k \) and \( \lambda \in X(k) \), define
  \[ \Delta^k(\lambda) = L(\lambda^0) \otimes \cdots \otimes L(\lambda^{k-1})(k-1) \otimes (\Delta(\sum_{j \geq k} p^{j-k} \chi_j))(k). \]
  Then \( \text{ch} \ \Delta^k(\lambda) = E^k_\lambda. \)

- \( \Delta^{k+1}(\lambda) \) is a quotient of \( \Delta^k_\lambda \) to get surjective maps of Dist(\( G \))-modules
  \[ \Delta(\lambda) = \Delta^0(\lambda) \rightarrow \cdots \rightarrow \Delta^k(\lambda) \rightarrow \cdots \rightarrow \Delta^\infty(\lambda) = L(\lambda). \]
For fixed $k$ and $\lambda \in X(k)$, define a highest weight module

$$M^k_{\lambda} = L(\lambda^0) \otimes \cdots \otimes L(\lambda^{k-1})^{(k-1)} \otimes (M(\sum_{j \geq k} p^{j-k} \lambda^j))^{(k)}.$$ 

Then $\text{ch} M^k_{\lambda} = F^k_{\lambda}$. Furthermore, we have surjective maps of $\text{Dist}(G)$-modules

$$M(\lambda) = M^0(\lambda) \to \cdots \to M^k(\lambda) \to \cdots \to M^\infty(\lambda) = L(\lambda).$$
THANK YOU!