Shapes of the irreducible morphisms and Auslander-Reiten Triangles in the stable category of modules over repetitive algebras

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Road map

1. Category of modules over repetitive algebras
2. Shapes of the irreducible morphisms
3. Shapes of Auslander-Reiten Triangles
4. Referencias

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Irreducibles and Auslander-Reiten Triangles
1. Category of modules over repetitive algebras

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4. Referencias
Let $A$ be a finite-dimensional $k$-algebra over field $k$.

For simplicity, we assume that $A$ is basic and $k$ is algebraically closed.

Denote by $D = \text{Hom}_k(-, k)$ the standard duality on $A$-mod.

Let us construct the repetitive algebra $\hat{A}$ of $A$ as proposed by D. Hughes and J. Waschbüsch (1983).

- The underlying vector space of repetitive algebra $\hat{A}$ is given by
  \[
  \hat{A} = (\bigoplus_{i \in \mathbb{Z}} A) \oplus (\bigoplus_{i \in \mathbb{Z}} DA),
  \]
  \[
  \hat{a} = (a_i, \varphi_i)_{i \in \mathbb{Z}} \text{ with } a_i \in A, \varphi_i \in DA \text{ and almost all } a_i, \varphi_i \text{ being zero.}
  \]

- The multiplication is defined by
  \[
  \hat{a} \cdot \hat{b} = (a_i, \varphi_i)_{i \in \mathbb{Z}} \cdot (b_i, \psi_i)_{i \in \mathbb{Z}} = (a_i b_i, a_i+1 \psi_i + \varphi_i b_i)_{i \in \mathbb{Z}}.
  \]
A \( \hat{A} \)-module \( M = (M_i, f_i)_{i \in \mathbb{Z}} \), where the \( M_i \) are \( A \)-modules, all but finitely many being zero (finitely generated left module), the \( f_i \) are \( A \)-homomorphims \( f_i : DA \otimes_A M_i \to M_{i+1} \), such that \( f_{i+1}(1 \otimes f_i) = 0 \) for all \( i \in \mathbb{Z} \).

Instead of \( M = (M_i, f_i)_{i \in \mathbb{Z}} \) we also write:

\[
M : \quad \cdots \longrightarrow M_{i-1} \xrightarrow{f_{i-1}} M_i \xrightarrow{f_i} M_{i+1} \longrightarrow \cdots
\]
A $\hat{A}$-module $M = (M_i, f_i)_{i \in \mathbb{Z}}$, where the $M_i$ are $A$-modules, all but finitely many being zero (finitely generated left module), the $f_i$ are $A$-homomorphims $f_i : DA \otimes_A M_i \longrightarrow M_{i+1}$, such that $f_{i+1}(1 \otimes f_i) = 0$ for all $i \in \mathbb{Z}$.

Instead of $M = (M_i, f_i)_{i \in \mathbb{Z}}$ we also write:

$$M : \quad \cdots \longrightarrow M_{i-1} \overset{f_{i-1}}{\longrightarrow} M_i \overset{f_i}{\longrightarrow} M_{i+1} \longrightarrow \cdots$$
A $\hat{A}$-homomorphism $h : M = (M_i, f_i)_{i \in \mathbb{Z}} \longrightarrow N = (N_i, g_i)_{i \in \mathbb{Z}}$ between $\hat{A}$-modules is a sequence $h = (h_i)_{i \in \mathbb{Z}}$ of $A$-homomorphims

$$DA \otimes_A M_i \xrightarrow{f_i} M_{i+1} \xrightarrow{h_i+1} N_i \xrightarrow{g_i} N_{i+1}.$$ 

Instead of $h = (h_i)_{i \in \mathbb{Z}} : M = (M_i, f_i)_{i \in \mathbb{Z}} \longrightarrow N = (N_i, g_i)_{i \in \mathbb{Z}}$ we also write:

$$M : \cdots \longrightarrow M_{i-1} \xrightarrow{f_{i-1}} M_i \xrightarrow{h_i} M_{i+1} \xrightarrow{h_{i+1}} \cdots$$

$$N : \cdots \longrightarrow N_{i-1} \xrightarrow{g_{i-1}} N_i \xrightarrow{g_i} N_{i+1} \xrightarrow{\cdots} \cdots.$$
A $\hat{A}$-homomorphism $h : M = (M_i, f_i)_{i \in \mathbb{Z}} \rightarrow N = (N_i, g_i)_{i \in \mathbb{Z}}$ between $\hat{A}$-modules is a sequence $h = (h_i)_{i \in \mathbb{Z}}$ of $A$-homomorphims

$$
\begin{align*}
DA \otimes_A M_i & \xrightarrow{f_i} M_{i+1} \\
1 \otimes h_i & \downarrow \quad h_{i+1} \\
DA \otimes_A N_i & \xrightarrow{g_i} N_{i+1}.
\end{align*}
$$

Instead of $h = (h_i)_{i \in \mathbb{Z}} : M = (M_i, f_i)_{i \in \mathbb{Z}} \rightarrow N = (N_i, g_i)_{i \in \mathbb{Z}}$ we also write:

$$
\begin{align*}
M : \quad & \cdots \xrightarrow{f_{i-1}} M_{i-1} \xrightarrow{f_i} M_i \xrightarrow{f_i} M_{i+1} \xrightarrow{f_i} \cdots \\
& \downarrow h_{i-1} \quad h_i \quad h_{i+1} \\
N : \quad & \cdots \xrightarrow{g_{i-1}} N_{i-1} \xrightarrow{g_i} N_i \xrightarrow{g_i} N_{i+1} \xrightarrow{g_i} \cdots.
\end{align*}
$$
We denoted by $\hat{A}$-mod the category of finitely generated left modules over the repetitive algebra $A$.

We denoted by $\hat{A}$-mod the stable category of $\hat{A}$-mod.
Section

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Definition

An $\widehat{A}$-homomorphism $h = (h_i)_{i \in \mathbb{Z}} : M = (M_i, f_i)_{i \in \mathbb{Z}} \rightarrow N = (N_i, g_i)_{i \in \mathbb{Z}}$:

1. is called smonic (resp. sepic) if all its components $h_i$ are split monomorphisms (resp. split epimorphisms) and

2. is called sirreducible if there is exactly one index $i_0$ such that $h_{i_0}$ is irreducible morphism and $h_i$ is a split epimorphism for $i < i_0$ and a split monomorphism for $i > i_0$. 
Shapes of the irreducible morphisms

Theorem (-, 2017, [2])

Let \( h : M = (M_i, f_i)_{i \in \mathbb{Z}} \rightarrow N = (N_i, g_i)_{i \in \mathbb{Z}} \) be an irreducible homomorphism in \( \hat{A} \)-mod. Then one of the following conditions holds:

1. \( h \) is a smonic morphism;
2. \( h \) is a sepic morphism;
3. \( h \) is a sirreducible morphism.
Irreducible smonic

\[
\begin{array}{cccccccc}
M_{a-1} & \xrightarrow{f_{a-1}} & M_a & \xrightarrow{f_a} & M_{a+1} & \cdots & \xrightarrow{f_{b-1}} & M_b & \xrightarrow{f_b} & M_{b+1} \\
\downarrow 1 & & \downarrow (1,0)^t & & \downarrow (1,0)^t & & \downarrow (1,0)^t & & \downarrow (1,0)^t \\
M_{a-1} & \xrightarrow{d_{a-1}} & M_a & \xrightarrow{d_a} & M_{a+1} & \cdots & \xrightarrow{d_{b-1}} & M_b & \xrightarrow{d_b} & M_{b+1} \\
\end{array}
\]

where \( h_{[a,b]} \) is the mono heart of \( h \).

For all \( i < a - 1 \) we have that \( d_i = f_i \) and \( d_{a-1} = (f_{a-1}, 0)^t \).

For \( a \leq i < b \),

\[
d_i = \begin{pmatrix}
  f_i & b_i \\
  0 & \overline{g_i}
\end{pmatrix}, \text{ with } b_i \neq 0 \text{ for all } a \leq i < b.
\]

For all \( i \geq b \),

\[
d_i = \begin{pmatrix}
  f_i & 0 \\
  0 & \overline{g_i}
\end{pmatrix}.
\]
Irreducible sepic

\[
\begin{array}{ccccccccc}
N_{a-1} \oplus M'_{a-1} & \xrightarrow{d_{a-1}} & N_a \oplus M'_{a} & \xrightarrow{d_a} & N_{a+1} \oplus M'_{a+1} & \cdots & & & \xrightarrow{d_b} & N_{b+1} \\
\downarrow (1,0) & & \downarrow (1,0) & & \downarrow (1,0) & & \downarrow (1,0) & & \downarrow 1 \\
\cdots N_{a-1} & \xrightarrow{g_{a-1}} & N_a & \xrightarrow{g_{a}} & N_{a+1} & \cdots & & & \xrightarrow{g_{b}} & N_{b+1}
\end{array}
\]

where \(h_{[a,b]}\) is the epi heart of \(h\).

For all \(i > b\), we have that \(d_i = g_i\) and \(d_b = (g_b, 0)\).

For \(a \leq i < b\),

\[
d_i = \begin{pmatrix} g_i & 0 \\ c_i & f'_i \end{pmatrix}, \text{ with } c_i \neq 0 \text{ for all } a \leq i < b.
\]

For all \(i < a\),

\[
d_i = \begin{pmatrix} g_i & 0 \\ 0 & f'_i \end{pmatrix}.
\]
Irreducible sirreducible (monomorphism)

\[ \cdots \rightarrow N_{k-2} \overset{g_{k-2}}{\rightarrow} N_{k-1} \overset{d_{k-1}}{\rightarrow} M_k \overset{f_k}{\rightarrow} M_{k+1} \overset{f_{k+1}}{\rightarrow} M_{k+2} \rightarrow \cdots \]

\[ \cdots \rightarrow N_{k-2} \overset{g_{k-2}}{\rightarrow} N_{k-1} \overset{d_{k-1}}{\rightarrow} M_k \overset{f_k}{\rightarrow} M_{k+1} \overset{f_{k+1}}{\rightarrow} M_{k+2} \rightarrow \cdots \]

where \( h_k \) is an irreducible \( A \)-monomorphism.

For \( i > k \),

\[ d_i = \begin{pmatrix} f_i & 0 \\ 0 & g_i \end{pmatrix}. \]
Irreducible sirreducible (epimorphism)

\[ \cdots \rightarrow N_{k-2} \oplus M'_{k-2} \overset{d_{k-2}}{\rightarrow} N_{k-1} \oplus M'_{k-1} \overset{d_{k-1}}{\rightarrow} M_k \overset{f_k}{\rightarrow} M_{k+1} \overset{f_{k+1}}{\rightarrow} M_{k+2} \rightarrow \cdots \]

\[ \downarrow (1,0) \]

\[ \cdots \rightarrow N_{k-2} \overset{g_{k-2}}{\rightarrow} N_{k-1} \overset{g_{k-2}}{\rightarrow} N_k \overset{d_k}{\rightarrow} M_k \overset{f_{k+1}}{\rightarrow} M_{k+2} \rightarrow \cdots \]

where \( h_k \) is an irreducible \( A \)-epimorphism.

For \( i < k \),

\[ d_i = \begin{pmatrix} g_i & 0 \\ 0 & f'_i \end{pmatrix} \]
### Proposition

Let \( h : M = (M_i, f_i)_{i \in \mathbb{Z}} \longrightarrow N = (N_i, g_i)_{i \in \mathbb{Z}} \) be a homomorphism in \( \hat{A}\)-mod, such that \( M \) and \( N \) have not projective summands and let \( h \) be its stable class in \( \hat{A}\)-mod. Then, \( h \) is split mono (resp. split epi) if and only if \( h \) is split mono (resp. split epi).

### Proposition

Let \( h : M = (M_i, f_i)_{i \in \mathbb{Z}} \longrightarrow N = (N_i, g_i)_{i \in \mathbb{Z}} \) be a homomorphism in \( \hat{A}\)-mod, such that \( M \) and \( N \) have not projective summands and let \( h \) be its stable class in \( \hat{A}\)-mod. Then, \( h \) is irreducible if and only if \( h \) is irreducible.
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Theorem

1. The category $\hat{A}\text{-mod}$ has almost split sequences (1983 D. Hughes and J. Waschbüsch).

2. The category $\hat{A}\text{-mod}$ is a Frobenius, and the category $\hat{A} - \text{mod}$ is triangulated (1988 D. Happel).
Theorem (-, Y. Calderón-Henao, and JA. Vélez-Marulanda, preprint, [1])

Let \( M = (M_i, f_i)_{i \in \mathbb{Z}} \xrightarrow{u} N = (N_i, g_i)_{i \in \mathbb{Z}} \xrightarrow{v} L = (L_i, l_i)_{i \in \mathbb{Z}} \xrightarrow{w} T(M) \) (1)
be an Auslander-Reiten triangle in \( \hat{A} \mod \). Then there exist an almost split sequence

\[
0 \rightarrow M \xrightarrow{u} N \oplus P \xrightarrow{v} L \xrightarrow{w} 0
\]

in \( \hat{A} \mod \), with \( P \) an \( \hat{A} \)–projective module, such that the triangle induce by this sequence is isomorphic to (1). If \( P \neq 0 \), then \( P \) is indecomposable, \( \text{rad}(P) \cong M \), and \( L \cong P / \text{soc}(P) \).
Theorem (-, Y. Calderón-Henao, and JA. Vélez-Marulanda, preprint, [1])

Let \( M = (M_i, f_i)_{i \in \mathbb{Z}} \xrightarrow{u} N = (N_i, g_i)_{i \in \mathbb{Z}} \xrightarrow{v} L = (L_i, l_i)_{i \in \mathbb{Z}} \xrightarrow{w} T(M) \) be an Auslander-Reiten triangle in \( \hat{A} - \text{mod} \). Then

1. If \( u \) is smonic, then \( v \) is sepic.
2. If \( u \) is sepic, then \( v \) is sirreducible.
3. If \( u \) is sirreducible, then \( v \) is smonic or sirreducible.
The quiver of a repetitive algebra

Theorem (1999 J. Schröer)

Let $Q$ be a finite quiver, and let $\rho$ be a set of relations for $Q$ which are either zero-relations or commutativity-relations such that $(Q, \rho)$ is locally bounded. Let $(\hat{Q}, \hat{\rho})$ be constructed as in (1999 J. Schröer). Then $k\hat{Q}/ <\hat{\rho}>$ is the repetitive algebra of $kQ/ <\rho>$.

Theorem (1991 C. M. Ringel and 1999 J. Schröer)

Let $A$ be a finite-dimensional $k$-algebra. Then

$A$ is gentle if and only if $\hat{A}$ is special biserial.
Example

Let $A_1$ be the finite dimensional algebra given by the quiver

$$Q: \begin{array}{c}
\bullet & \xleftarrow{\alpha} & \bullet & \xleftarrow{\beta} & \bullet
\end{array}. \ 	ext{The radical series of the indecomposables projective,}
\text{injective and simples left } A_1 \text{-modules are given as follows:}

$$

P_1 = 1, \quad S_2 = 2, \quad S_3 = 3, \quad P_2 = \begin{array}{c} 2 \\ 1 \end{array}, \quad P_3 = \begin{array}{c} 3 \\ 2 \\ 1 \end{array}, \quad I_2 = \begin{array}{c} 3 \\ 2 \end{array}

$$

The Auslander-Reiten quiver of $A_1$ is the given as follows:

![Auslander-Reiten quiver](image)
Recall that $\hat{Q}$ is given by

\[
\begin{array}{ccc}
1_{z+1} & \xrightarrow{\alpha_{z+1}} & 2_{z+1} \\
& q_{z+1} &
2_z & \xleftarrow{\alpha_z} & 3_z \\
1_z & \xrightarrow{q_z} & 2_{z-1} & \xleftarrow{\alpha_{z-1}} & 3_{z-1}
\end{array}
\]
\begin{align*}
P_{1z} &= \begin{array}{c}
1_z \\
\downarrow \\
3_{z-1}
\end{array} = I_{1z-1}, &
P_{2z} &= \begin{array}{c}
2_z \\
\downarrow \\
1_z
\end{array} = I_{2z-1} \text{ and } &
P_{3z} &= \begin{array}{c}
3_z \\
\downarrow \\
2_z
\end{array} = I_{3z-1}
\end{align*}
The stable Auslander-Reiten quiver of $\widehat{A}_1$ is given by

```
\begin{align*}
2_z & \xleftarrow{1_z} 3_z & 1_{z+1} & 2_{z+1} & 3_{z+1} \\
1_z & \xleftarrow{2_z} 2_z & 3_z & 1_{z+1} & 2_{z+1} \\
3_{z-1} & \xleftarrow{2_z} 1_z & 3_z & 1_{z+1} & 3_z \\
& \xleftarrow{1_z} 2_z & 3_z & 1_{z+1} & 2_{z+1} \\
& 1_z & 2_z & 3_z & 1_{z+1} & 2_{z+1}
\end{align*}
```
Auslander-Reiten triangle in $\mathring{A}_1$-mod

\[
\begin{array}{c}
2_z \\
1_z \\
3_z - 1 \\
\downarrow \\
2_z \\
1_z \\
3_z \\
\downarrow \\
2_z \\
1_z \\
\downarrow \\
\end{array}
\xrightarrow{\quad -z \quad} \xrightarrow{\quad -z + 1 \quad}
\begin{array}{c}
2 \\
1 \\
3 \\
\downarrow \\
2 \\
1 \\
\downarrow \\
\end{array}
\xrightarrow{\quad 0 \quad} \xrightarrow{\quad 0 \quad} \xrightarrow{\quad 0 \quad}
\begin{array}{c}
2 \\
1 \\
3 \\
\downarrow \\
2 \\
1 \\
\downarrow \\
\end{array}
\xrightarrow{\quad 0 \quad} \xrightarrow{\quad 0 \quad} \xrightarrow{\quad 0 \quad}
\]

sepic

sirreducible
Auslander-Reiten triangle in $\widehat{A_1}\text{-mod}$

```
3z  \rightarrow  0  \rightarrow  3  \rightarrow  0  \rightarrow  0  \rightarrow  \\
\downarrow  \downarrow  \downarrow  \downarrow  \downarrow  \\
1_{z+1}  \rightarrow  1  \rightarrow  3  \rightarrow  0  \rightarrow  0  \rightarrow  \\
\downarrow  \downarrow  \downarrow  \downarrow  \downarrow  \\
3z  \rightarrow  0  \rightarrow  3  \rightarrow  0  \rightarrow  0  \rightarrow  \\
\downarrow  \downarrow  \downarrow  \downarrow  \downarrow  \\
1_{z+1}  \rightarrow  1  \rightarrow  0  \rightarrow  0  \rightarrow  0  \rightarrow  \\
\downarrow  \downarrow  \downarrow  \downarrow  \downarrow  \\
```

$smonic$

$sepic$
Auslander-Reiten triangle in $\widehat{A}_1$-mod

\[ \begin{array}{c}
- z - 1 \\
0 \\
1 \\
0 \\
0 \\
\end{array} \]

\[ \begin{array}{c}
0 \\
1 \\
0 \\
0 \\
\end{array} \]

\[ \begin{array}{c}
0 \\
2 \\
0 \\
0 \\
\end{array} \]

$s_{irreducible}$

$s_{irreducible}$
Example

Let $A_2 = kQ/I$ be the finite dimensional algebra given by the quiver $Q := a \xleftarrow{c} 0 \xrightarrow{b} 1$, where $I = \langle a^2, bc, cb \rangle$. We have the quiver $\hat{Q}$ is given by
Auslander-Reiten triangle in $\widehat{A}_2\text{-mod}$

```
\begin{array}{ccc}
0_z & 0_z & 0_z \\
0_z & 1_z & 1_{z-1} \\
1_z & 0_{z-1} & 1_{z-1} \\
0_{z-1} & 0_{z-1} & 1_{z-1} \\
0_{z-1} & 1_{z-1} & 1_{z-2} \\
\end{array}
```

```
\begin{array}{ccc}
-z & -z + 1 & -z + 2 \\
0 & 0 & 1 \\
1 & 0 & 1 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
\end{array}
```

sirreducible

sirreducible

Irreducibles and Auslander-Reiten Triangles
Auslander-Reiten triangle in $\widehat{A}_2\text{-mod}$

- $z$
- $z + 1$

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Irreducibles and Auslander-Reiten Triangles

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Thanks
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4. Referencias
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