CMness is determined by inertia groups

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G ⊂ S_n acts on \( \mathbb{Z}[x_1, \ldots, x_n] \). Problem: describe invariant subring.

**Theorem (Fundamental Theorem on Symmetric Polynomials)**

*If G = S_n, then*

\[
\mathbb{Z}[x_1, \ldots, x_n]^G = \mathbb{Z}[\sigma_1, \ldots, \sigma_n],
\]

*where* \( \sigma_1 = \sum_i x_i, \sigma_2 = \sum_{i<j} x_i x_j, \) *etc.*

What if \( G \nsubseteq S_n \)?
Permutation invariants

Over \( \mathbb{Q} \):

**Theorem (Kronecker 1881)**

\[ \mathbb{Q}[x_1, \ldots, x_n]^G \text{ is a free module over } \mathbb{Q}[\sigma_1, \ldots, \sigma_n]. \]

Kronecker’s contribution is not well-known, but a modern invariant theorist would see this as an immediate consequence of the Hochster-Eagon theorem.
Example

\[ G = \langle (1234) \rangle \subset S_4, \text{ acting on } \mathbb{Q}[x, y, z, w]. \]

\[ g_0 = 1 \]
\[ g_2 = xz + yw \]
\[ g_3 = x^2y + y^2z + \ldots \]
\[ g_{4a} = x^2yz + y^2zw + \ldots \]
\[ g_{4b} = xy^2z + yz^2w + \ldots \]
\[ g_5 = x^2y^2z + y^2z^2w + \ldots \]

is a basis over \( \mathbb{Q}[\sigma_1, \ldots, \sigma_4] \).

\[ x^3y^2z + y^3z^2w + \cdots = \frac{1}{2} \sigma_3 g_3 - \frac{1}{2} \sigma_2 g_{4b} + \frac{1}{2} \sigma_1 g_5 \]
Permutation invariants

Statement fails over \( \mathbb{Z} \).

\[
x^3y^2z + y^3z^2w + \cdots = \frac{1}{2}\sigma_3g_3 - \frac{1}{2}\sigma_2g_4b + \frac{1}{2}\sigma_1g_5
\]

**Problem**

*For which \( G \subset S_n \) does the statement of Kronecker’s theorem hold over \( \mathbb{Z} \)?*

**Equivalent to:**

*For which \( G \subset S_n \) is \( k[x_1, \ldots, x_n]^G \) a Cohen-Macaulay ring for any field \( k \)?
Permutation invariants

Known as of 2016:

- If $k[x_1, \ldots, x_n]^G$ is CM and char $k = p$, then $G$ is generated by bireflections and $p'$-elements. (Gordeev & Kemper '03)
- If $G =$
  - $S_n$ (classical)
  - $A_n$ (classical)
  - $S_n \cong S_n \times S_n \subset S_{2n}$ (Reiner ’90 / ’03), or
  - $S_2 \wr S_n \subset S_{2n}$ (Hersh ’03),
  
then $k[x_1, \ldots, x_n]^G$ is CM regardless of $k$. 
Permutation invariants

Let $k[x] := k[x_1, \ldots, x_n]$.

**Theorem (BBS ’17)**

If $G \subset S_n$ is generated by transpositions, double transpositions, and 3-cycles, then $k[x]^G$ is Cohen-Macaulay regardless of $k$.

**Theorem (BBS - Sophie Marques ’18)**

The converse is also true.

Compare: Chevalley-Shepard-Todd [cf. Ellen’s talk]
Story of the “only-if” direction

The “if” direction used Stanley-Reisner theory to translate an orbifold result of Christian Lange ("topological Chevalley-Shephard-Todd") into a CMness result for a certain Stanley-Reisner ring $k[\Delta/G]$, and then work of Garsia and Stanton ’84 to translate this into the desired result for $k[\mathfrak{x}]^G$.

By a topological argument, for the Stanley-Reisner ring $k[\Delta/G]$, the “only-if” held too. However, the arguments of Garsia-Stanton do not allow one to transfer this conclusion back to $k[\mathfrak{x}]^G$.

After I defended, Sophie Marques proposed to transfer the argument, rather than the conclusion, from $k[\Delta]^G$ to $k[\mathfrak{x}]^G$.

This necessitated a search for a commutative-algebraic fact to replace each topological fact we used.
Let $X$ be a Hausdorff topological space carrying an action by a finite group $G$. Let $x \in X$. Let $G_x$ be the stabilizer of $x$ for the action of $G$. Let $X/G$ be the topological quotient, and let $\overline{x}$ be the image of $x$ in $X/G$.

**Theorem (slice theorem for finite groups)**

There is a neighborhood $U$ of $x$, invariant under $G_x$, such that $U/G_x$ is homeomorphic to a neighborhood of $\overline{x}$ in $X/G$.

The name “slice theorem” comes from an analogous result for compact Lie groups.
Local structure in a quotient – slice theorem

What is the commutative-algebraic analogue?

There is an algebraic group analogue to the Lie group slice theorem called *Luna’s étale slice theorem*, but for finite $G$, there is something much more general.

Let $A$ be a ring with an action of a finite group $G$.

- $x \in X$ becomes $\mathfrak{P} \triangleleft A$.
- $X/G$ becomes $A^G$.
- $\bar{x} \in X/G$ becomes $p = \mathfrak{P} \cap A^G$.
- $G_x$ becomes $l_G(\mathfrak{P}) := \{ g \in G : a - ga \in \mathfrak{P}, \ \forall a \in A \}$. (Not $D_G(\mathfrak{P})$!)
- The appropriate analogue for the sufficiently small neighborhood of $\bar{x}$ in $X/G$ is the *strict henselization* of $A^G$ at $p$. 
Let $C$ be a (commutative, unital) ring. Let $p$ be a prime ideal of $C$. The *strict henselization* of $C$ at $p$ is a local ring $C_{p}^{hs}$ together with a local map $C_p \rightarrow C_{p}^{hs}$ with the following properties:

1. $C_{p}^{hs}$ is a henselian ring.
2. $\kappa(C_{p}^{hs})$ is the separable closure of $\kappa(C_p)$.
3. $C_p$ and $C_{p}^{hs}$ are simultaneously noetherian (resp. CM).
4. $C_p \rightarrow C_{p}^{hs}$ is faithfully flat of relative dimension zero.

$C_{p}^{hs}$ is universal with respect to 1 and 2. It should be viewed as a “very small neighborhood of $p$ in $C$.”
Local structure in a quotient – slice theorem

Let $A$ be a ring with an action by a finite group $G$. Let $p$ be a prime of $A^G$. Let $C_p^{hs}$ be the strict henselization of $A^G$ at $p$. Define

$$A_p^{hs} := A \otimes_{A^G} C_p^{hs}$$

Note $G$ acts on $A_p^{hs}$ through its action on $A$.

Let $\mathfrak{P}$ be a prime of $A$ lying over $p$ and let $\mathfrak{Q}$ be a prime of $A_p^{hs}$ pulling back to $\mathfrak{P}$. Recall $I_G(\mathfrak{P}) := \{g \in G : a - ga \in \mathfrak{P}, \forall a \in A\}$. (Fact: $I_G(\mathfrak{Q}) = I_G(\mathfrak{P})$.)

**Theorem (Raynaud ’70)**

There is a ring isomorphism $(A_p^{hs})_{\mathfrak{Q}}^{I_G(\mathfrak{P})} \cong C_p^{hs}$.

This is the commutative-algebraic analogue!
Corollary (BBS - Marques ’18)

Assume $A^G$ is noetherian. Then TFAE:

1. $A^G$ is CM.

2. For every prime $p$ of $A^G$ and every $Q$ of $A_p^{hs}$ pulling back to a $P$ of $A$ lying over $p$,

   $$(A_p^{hs})_{Q}^{I_G}(P)$$

   is CM.

3. For every maximal $p$ of $A^G$, there is some $Q$ of $A_p^{hs}$ pulling back to a $P$ of $A$ lying over $p$, such that

   $$(A_p^{hs})_{Q}^{I_G}(P)$$

   is CM.
Back to the permutation group context. Let $k[x] = \mathbb{F}_p[x_1, \ldots, x_n]$, for some prime $p$ to be determined later.

Let $N$ be the subgroup of $G$ generated by transpositions, double transpositions, and 3-cycles.

Goal: prove that, if $N \subseteq G$, for the right choice of $p$, $k[x]^G$ is not CM.

By above, it suffices to find, when $N \subseteq G$, a $p \triangleleft k[x]^G$ such that the corresponding $C_{p}^{hs}$ is not CM.
Note that $G/N$ acts on $k[x]^N$.

**Theorem (BBS - Marques ’18)**

*If there is a prime $\mathfrak{p}$ of $k[x]^N$ whose inertia group $I_{G/N}(\mathfrak{p})$ is a $p$-group, then $k[x]^G$ is not CM.*

(Recall $k = \mathbb{F}_p$.)

The main ingredients of the proof are:

- the above result which says that CMness at $\mathfrak{p} \cap k[x]^G$ only depends on the action of $I_{G/N}(\mathfrak{p})$ on the appropriate strict henselization.
- a theorem of Lorenz and Pathak ’01 which shows that such $I_{G/N}(\mathfrak{p})$ obstructs CMness.

It also uses the “if” direction to conclude that $k[x]^N$ is CM. The invocation of Lorenz and Pathak needs this.
So the problem is reduced to finding a prime number $p$ and a prime ideal $\mathfrak{P}$ of $k[x]^N$ such that $I_{G/N}(\mathfrak{P})$ is a $p$-group, when $N \subset G$.

Let $\Pi_n$ be the poset of partitions of $[n]$, ordered by refinement.

Each $\pi \in \Pi_n$ corresponds to the ideal $\mathfrak{P}_\pi^*$ of $k[x]$ generated by $x_i - x_j$ for each pair $i, j$ in the same block of $\pi$. (Cf. the braid arrangement.)

Let $G^B_\pi$ be the blockwise stabilizer of $\pi$ in $G$, and let $G^B_\pi N/N$ be its image in $G/N$. If $\mathfrak{P}_\pi = \mathfrak{P}_\pi^* \cap k[x]^N$, one can show that

$$I_{G/N}(\mathfrak{P}_\pi) = G^B_\pi N/N.$$
Permutation invariants - “only-if” direction

So we just need to find \( \pi \) such that \( G^B_\pi N/N \) is a \( p \)-group.

Consider the map

\[
\varphi : G \rightarrow \Pi_n
\]

that sends a permutation \( g \) to the decomposition of \([n]\) into orbits of \( g \).

**Proposition (BBS '17)**

If \( g \in G \setminus N \) is such that \( \pi = \varphi(g) \) is minimal in \( \varphi(G \setminus N) \), then \( G^B_\pi N/N \) has prime order (and is generated by the image of \( g \)).

If \( N \subsetneq G \), \( G \setminus N \) is nonempty, so such \( g \) exists, and fixing \( p \) as the order of \( G^B_\pi N/N \), we find that \( k[x]^G \) is not CM.
Is there a uniform proof of Lange’s theorem?

Is there a purely algebraic proof of the “if” direction?

Given $G$ not generated by transpositions, double transpositions, and three-cycles, find all the “bad” primes?
Thank you!