Tame quivers have finitely many m-maximal green sequences

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Tame quivers have finitely many m-maximal green sequences.
1 Background
   - Tame Quivers
   - Silting Objects
   - m-maximal green sequences

2 Our Result
   - Theorem
   - The proof
Outline

1 Background
   - Tame Quivers
   - Silting Objects
   - m-maximal green sequences

2 Our Result
   - Theorem
   - The proof
A **tame quiver** is a quiver such that its path algebra is a tame algebra.
Tame Quivers

Tame Quivers

Definition

A *tame quiver* is a quiver such that its path algebra is a tame algebra. A *tame algebra* is a $k$-algebra such that for each dimension there are finitely many 1-parameter families that parametrize all but finitely many indecomposable modules of the algebra.
Tame Quivers

Definition

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A *tame algebra* is a $k$-algebra such that for each dimension there are finitely many 1-parameter families that parametrize all but finitely many indecomposable modules of the algebra.

Example

Here are all the (connected) tame quivers, $\tilde{A}_n, \tilde{D}_n, \tilde{E}_6, \tilde{E}_7, \tilde{E}_8$. 

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Tame Quivers

Auslander-Reiten Quivers

Theorem

The Auslander-Reiten quiver of a tame path algebra consists of three parts, the preprojectives, the preinjectives and the regulars.
Here are some basic properties of preprojective and preinjective components of AR quivers of basic tame hereditary algebras.

1. The AR quiver of \( kQ \) has one preprojective component which looks like \( \mathbb{N}Q^{op} \).
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3. All preprojective and preinjective modules in $kQ$ are rigid.
Here are some basic properties of preprojective and preinjective components of AR quivers of basic tame hereditary algebras.

1. The AR quiver of $kQ$ has one preprojective component which looks like $\mathbb{N}Q^{\text{op}}$.
2. The AR quiver of $kQ$ has one preinjective component which looks like $-\mathbb{N}Q^{\text{op}}$.
3. All preprojective and preinjective modules in $kQ$ are rigid.
4. All but finitely many preprojectives and preinjectives are sincere.
Here are some basic properties of regular components of AR quivers of basic tame hereditary algebras.

1. There are infinitely many regular components, all of which are standard tubes $\mathbb{Z}A_\infty/(\tau^k)$. 
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1. There are infinitely many regular components, all of which are standard tubes $\mathbb{Z}A_\infty/(\tau^k)$.

2. All but at most three tubes have $k = 1$. In this case we consider the component homogeneous.

3. All elements in a homogeneous tube are non-rigid, hence they and their shifts can not be summands of any silting object.
Here are some basic properties of regular components of AR quivers of basic tame hereditary algebras.

4 In a nonhomogeneous component $\mathbb{Z}A_\infty/(\tau^k)$ only indecomposables with quasi-length less than $k$ are rigid. In other words there are only finitely many rigid indecomposables in any nonhomogeneous component.
Here are some basic properties of regular components of AR quivers of basic tame hereditary algebras.

4. In a nonhomogeneous component $\mathbb{Z}A_{\infty}/(\tau^k)$ only indecomposables with quasi-length less than $k$ are rigid. In other words there are only finitely many rigid indecomposables in any nonhomogeneous component.

5. Only finitely many regular indecomposable modules are rigid. Hence only finitely many regular indecomposables and their shifts can appear in an $m$-maximal green sequence.
Tame Quivers

Standard Stable Tubes

This is a standard stable tube

with rank 3. $M_{ik}$ is rigid iff $k \leq 2$. 

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Tame quivers have finitely many m-maximal green sequences
This is a standard stable tube with rank 3. $M_{ik}$ is rigid iff $k \leq 2$. $M_{i+k-1}$

Here $M_{ik} = \ldots$. We define the quasi-length of $M_{ik}$ as $k$. $M_{i+1}$

$M_{i}$

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Now let’s see a homogeneous tube.
Now let's see a homogeneous tube.

\[
\cdots
\]

\[
\xymatrix{ & M_3 \\
M_2 \\
M }
\]

Note that nothing in this tube is rigid.
Now let’s see a homogeneous tube.

\[ \cdots \]

\[
\begin{array}{c}
\nearrow & \searrow \\
M_3 & \\
\nearrow & \searrow \\
M_2 & \\
\nearrow & \searrow \\
M & \\
\end{array}
\]

Note that nothing in this tube is rigid.

Here \( M_k = \cdots \)
So here is what an AR quiver of a tame path algebra looks like. In this example the quiver is $1 \to 5 \leftarrow 3$. 

```
2
↓
1
```

```
4
```

Tame quivers have finitely many m-maximal green sequences.
Tame Quivers

Example: $\tilde{D}_4$-preprojectives

Here is the preprojective component, $\mathcal{P}$. 
Tame Quivers

Example: $\tilde{D}_4$-preprojectives

Here is the preprojective component, $\mathcal{P}$.

$$
P_1 \xrightarrow{\tau^{-1}} P_1 \rightarrow \cdots

P_2 \xrightarrow{\tau^{-1}} P_2 \rightarrow \cdots

P_3 \xrightarrow{\tau^{-1}} P_3 \rightarrow \cdots

P_4 \xrightarrow{\tau^{-1}} P_4 \rightarrow \cdots

\tau^{-1}P_5 \rightarrow \cdots

\tau^{-2}P_5 \rightarrow \cdots

\tau^{-1}P_5 \rightarrow \cdots

\tau^{-2}P_5 \rightarrow \cdots

$$

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Here is the preinjective component, $Q$. 
Tame Quivers

Example: $\tilde{D}_4$-preinjectives

Here is the preinjective component, $Q$.

\[ \cdots \xrightarrow{\tau l_1} l_1 \xrightarrow{\tau l_2} l_2 \xrightarrow{\tau l_3} l_3 \xrightarrow{\tau l_4} l_4 \xrightarrow{\tau l_5} l_5 \cdots \]

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Tame quivers have finitely many $m$-maximal green sequences
Here are the regular components. There are infinitely many homogeneous tubes and 3 nonhomogeneous ones. All objects in the homogeneous ones are non-rigid. The quasi-simple in the homogeneous tubes has dimension vector is $(1,1,1,1,2)$. The quasi-simples in the three nonhomogeneous tubes have dimension vectors $(1,1,0,0,1)$ and $(0,0,1,1,1)$, $(1,0,1,0,1)$ and $(0,1,0,1,1)$, $(1,0,0,1,1)$ and $(0,1,1,0,1)$ respectively.
For a tame quiver $Q$ there are infinitely many components of $D^b(kQ)$ consisting of shifts of preprojectives and preinjectives that are in the form $\mathbb{Z}Q^{op}$. Let’s label these components transjective. The transjective component containing $\Lambda[m]$ is labelled $\mathcal{P}_m$. 

Tame Quivers
AR quivers of Bounded Derived Categories

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Tame quivers have finitely many m-maximal green sequences
For a tame quiver $Q$ there are infinitely many components of $D^b(kQ)$ consisting of shifts of preprojectives and preinjectives that are in the form $\mathbb{Z} Q^{op}$. Let’s label these components transjective. The transjective component containing $\Lambda[m]$ is labelled $\mathcal{P}_m$. There are also infinitely many regular components. There are at most 3 nonhomogeneous tubes in $\text{mod} kQ[m]$ for any $m$. There are also infinitely many homogeneous tubes in $\text{mod} kQ[m]$ for any $m$. However since nothing in a homogeneous tube is rigid they don’t affect our problem.
Outline

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- Tame Quivers
- Silting Objects
- m-maximal green sequences

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Silting Objects
Silting Objects

Definition

Let $\Lambda$ be an algebra with $n$ primitive idempotents. A silting object $T$ of $D^b(\Lambda)$ is an object such that $T$ has $n$ direct summands and $(T, T[m]) = 0$ for all $m > 0$. A pre-silting object is an object that only has to satisfy the second condition.
Silting Objects

Ex: $D^b(A_3)$


$\Gamma[i]$ is a silting object for any $i$.

$T_1 = P_3[1] \oplus P_1[1] \oplus I_1[1]$ is also a silting object.

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\( \Lambda[i] \) is a silting object for any \( i \). \( T_1 = P_3[1] \oplus P_1[1] \oplus I_1[1] \) is also a silting object.
Definition

Let $\mathcal{C}$ be a category and $\mathcal{X}$ be one of its subcategories. If $M \in \text{Ob}\mathcal{C}$, $N \in \text{Ob}\mathcal{X}$, a morphism $f \in \text{Hom}_\mathcal{C}(M, N)$ is a \textit{minimal left-$\mathcal{X}$ approximation} if for any $g \in \text{End}_\mathcal{C} N$ such that $g \circ f = f \circ g$ is an isomorphism and for any $N' \in \text{Ob}\mathcal{X}$ for any $q \in \text{Hom}_\mathcal{C}(M, N')$ we have $q$ factors through $f$. 

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Definition

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\[
\begin{array}{ccc}
M & \xrightarrow{f} & N \\
\downarrow q & & \downarrow l \\
N' & \downarrow & \\
\end{array}
\]
Definition

Let \( \mathcal{C} \) be a category and \( \mathcal{X} \) be one of its subcategories. If \( N \in \text{Ob}\mathcal{C}, M \in \text{Ob}\mathcal{X}, \) A morphism \( f \in \text{Hom}_\mathcal{C}(M, N) \) is a minimal right-\( \mathcal{X} \) approximation if for any \( g \in \text{End}_\mathcal{C}M \) such that \( f \circ g = f \) \( g \) is an isomorphism and for any \( M' \in \text{Ob}\mathcal{X} \) for any \( q \in \text{Hom}_\mathcal{C}(M', N) \) we have \( q \) factors through \( f \).
Definition

Let $\mathcal{C}$ be a category and $\mathcal{X}$ be one of its subcategories. If $N \in \text{Ob}\mathcal{C}$, $M \in \text{Ob}\mathcal{X}$, A morphism $f \in \text{Hom}_\mathcal{C}(M, N)$ is a minimal right-$\mathcal{X}$ approximation if for any $g \in \text{End}_\mathcal{C}M$ such that $f \circ g = f \circ g$ is an isomorphism and for any $M' \in \text{Ob}\mathcal{X}$ for any $q \in \text{Hom}_\mathcal{C}(M', N)$ we have $q$ factors through $f$.
Definition

A *forward mutation* on the direct summand $T_i$ of the silting object $T$ is $T'_i \oplus (T/T_i)$ where $T'_i$ is the cone/homotopy cokernel of the minimal left-$\text{add}(T/T_i)$ approximation of $T_i$. 

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Tame quivers have finitely many m-maximal green sequences
Silting Objects

Mutations

Definition

A forward mutation on the direct summand $T_i$ of the silting object $T$ is $T'_i \oplus (T/T_i)$ where $T'_i$ is the cone/homotopy cokernel of the minimal left-\text{-}\text{add}(T/T_i)$ approximation of $T_i$.

A backward mutation on the direct summand $T_i$ of the silting object $T$ is $T'_i \oplus (T/T_i)$ where $T'_i$ is homotopy kernel/ [-1] of the cone/ of the minimal right-\text{-}\text{add}(T/T_i)$ approximation of $T_i$. 
Silting Objects

Ex: $D^b(A_3)$

\[
\begin{align*}
I_1[-1] & \quad P_1 & \quad P_3[1] & \quad S_2[1] & \quad I_1[1] \\
\quad P_2 & \quad I_2[1] & \quad P_2[1] & \quad I_2[1] & \quad I_1[1] \\
\quad P_3 & \quad S_2 & \quad I_1 & \quad P_1[1] & \quad P_3[2]
\end{align*}
\]

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Tame quivers have finitely many m-maximal green sequences
Silting Objects
Ex: $D^b(A_3)$

$\Lambda$ is a silting object. When we do a forward mutation at $P_3$ we get $T' = S_2 \oplus P_2 \oplus P_1$. When we do a forward mutation at $P_1$ now we get $T'' = S_2 \oplus P_2 \oplus P_1[1]$. When we do another forward mutation at $P_2$ we get $T''' = S_2 \oplus P_3[1] \oplus P_1[1]$. 

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m-maximal green sequences

Definition

An *m-maximal green sequence* is a finite sequence of forward mutations from $\Lambda$ to $\Lambda[m]$. 
m-maximal green sequences

Example

\[ I_1[-1] \rightarrow P_2 \rightarrow P_1 \rightarrow P_3[1] \overset{l_1}{\rightarrow} I_1 \rightarrow P_1[1] \rightarrow P_2[1] \rightarrow I_2[1] \rightarrow S_2[1] \rightarrow P_3[2] \rightarrow I_1[1] \]

So \((P_1, P_2, P_3, P_1[1], P_2[1], P_3[2])\) is a 2-maximal green sequence.
m-maximal green sequences

Example

So \((P_1, P_2, P_3, P_1[1], P_2[1], P_3[1])\) is a 2-maximal green sequence.
1 Background
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Theorem

Tame quivers accept finitely many $m$-maximal green sequences.
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The basic idea here is that if only finitely many indecomposable summands can appear in any $m$-maximal green sequence then only finitely many silting objects can appear in any $m$-maximal green sequence.
The basic idea here is that if only finitely many indecomposable summands can appear in any $m$-maximal green sequence then only finitely many silting objects can appear in any $m$-maximal green sequence. When that happens only finitely many $m$-maximal green sequences can exist because a green sequence can not repeat silting objects due to Theorem 2.11 in [1].
Hence the problem has been reduced to the problem of whether there are only finitely many indecomposable summands of silting objects in any $m$-maximal green sequence.
Hence the problem has been reduced to the problem of whether there are only finitely many indecomposable summands of silting objects in any $m$-maximal green sequence. There are only two kinds of indecomposable summands, namely transjectives of the form $\tau^i P_j[k]$ and regulars (i.e. shifts of regular modules).
However there are only finitely many nonhomogeneous regular components between $\Lambda$ and $\Lambda[m]$ and each of them only have finitely many rigid indecomposable. Furthermore homogeneous regular components do not have any rigid objects. Hence only finitely many rigid regular indecomposable summands can appear in any $m$-maximal green sequence.
Hence the problem is really only about transjectives.
Hence the problem is really only about transjectives. We will follow the approach of Brustle-Dupont-Perotin here. We first prove that for a tame quiver with $n$ vertices there are at most $n - 2$ regulars in a silting object.
Hence the problem is really only about transjectives. We will follow the approach of Brustle-Dupont-Perotin here. We first prove that for a tame quiver with $n$ vertices there are at most $n - 2$ regulars in a silting object. So here is our first lemma.

**Lemma**

1. If $Q$ is a tame quiver with $n$ vertices there are at most $n - 2$ regular indecomposable summands in a silting object of $D^b(kQ)$. 
Using properties of tame quivers it is easy to see using a type by type argument that Lemma 1 can be reduced to Lemma 2.
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Lemma

If $Q$ is a tame quiver with $n$ vertices and $R$ is one of the nonhomogeneous regular components of $\text{mod} kQ$ with $k$ quasi-simples. Then at most $k - 1$ summands in $\bigcup R[m]$ of may appear in any silting object in $D^b(kQ)$. 
If \( \{ M_i \}_{i \in I} \) are a family of indecomposable modules of \( kQ \) and \( \prod_{i \in I} M_i[n_i] \) is not pre-silting for any \( \{ n_i \}_{i \in I} \) we say that \( \{ M_i \}_{i \in I} \) is **silting-incompatible**. Otherwise we say that it is **silting-compatible**.
Now we need two more short lemmas, Lemma 3 and Lemma 4, to prove Lemma 2.

Lemma 3

If $M$ and $N$ are regular modules in a nonhomogeneous tube in the Auslander-Reiten quiver of $kQ$. If $\text{Hom}(M, N) \neq 0$ and $\text{Ext}^1(N, M) \neq 0$, then $M$ and $N$ are silting-incompatible.
Now we need two more short lemmas, Lemma 3 and Lemma 4, to prove Lemma 2.

**Lemma**

3. If $M$ and $N$ are regular modules in a nonhomogeneous tube in the Auslander-Reiten quiver of $kQ$. If $\text{Hom}(M, N) \neq 0$ and $\text{Ext}^1(N, M) \neq 0$, then $M$ and $N$ are silting-incompatible.

4. If $M_1, \cdots, M_k$ are regular modules in a nonhomogeneous tube in the Auslander-Reiten quiver of $kQ$. If $\text{Ext}^1(M_i, M_{i+1}) \neq 0$ for any $1 \leq i < k$ and $\text{Ext}^1(M_k, M_1) \neq 0$, then $\{M_i\}$ is silting-incompatible.
Proof.

For Lemma 3 since $\text{Hom}(M, N) \neq 0$, $\text{Ext}^{i-j}(M[i], N[j]) \neq 0$ if $i > j$. Since $\text{Ext}^1(N, M) \neq 0$ $\text{Ext}^{j-i+1}(N[j], M[i]) \neq 0$ if $i \leq j$. Hence $M[i] \oplus N[j]$ is not pre-silting for any arbitrary $i$ and $j$.

For Lemma 4 for arbitrary $n_1, \cdots n_k$ use the argument above it is easy to see that if $\bigoplus_{i=1}^k M_i[n_i]$ is pre-silting, then $n_2 > n_1$, $n_3 > n_2$, $\cdots$, $n_1 > n_k$ which is impossible. Hence $\{M_i\}$ is silting-incompatible.
Now let’s prove Lemma 2.

Proof.
If we assume that the conclusion in Lemma 2 is wrong we will reach a silting-incompatible scenario in either Lemma 3 or Lemma 4. Hence Lemma 2 is proven.
To make arguments easier we define the transjective degree of the transjective object $\tau^i P_j[k]$ to be $i$. After proving Lemma 2 we can prove that any mutation that changes components can only change the transjective degree of a transjective object by some bounded amount. There are only finitely many possible green mutations that change components in any $m$-maximal green sequence. We can prove that any transjective indecomposable summand allowed in an $m$-maximal green sequence has bounded transjective degree.
Lemma

([2], Lemma 10.1) Let $H$ be a representation-infinite connected hereditary algebra. Then there exists $N \geq 0$ such that for any $k \geq N$, for any projective $H$-module $P$, the $H$-modules $\tau^{-k}P$ and $\tau^{k}P[1]$ are sincere.
The proof
Basic ideas-The degree argument

**Lemma**

([2], Lemma 10.1) Let $H$ be a representation-infinite connected hereditary algebra. Then there exists $N \geq 0$ such that for any $k \geq N$, for any projective $H$-module $P$, the $H$-modules $\tau^{-k}P$ and $\tau^kP[1]$ are sincere.

**Lemma**

([2]) Let $Q$ be a tame quiver and $M_1, M_2$ two transjective modules of $kQ$. If $\{M_1, M_2\}$ is silting-compatible, then $|\deg(M_1) - \deg(M_2)| \leq N$
Proof.

If $k - l > N$ we need to prove that $\tau^k P_a$ and $\tau^l P_b$ are silting-incompatible. If $i \leq j$

$\text{Ext}^{j - i + 1}(\tau^l P_b[j], \tau^k P_a[i]) = \text{Ext}^1(\tau^l P_b, \tau^k P_a) = \text{Hom}(\tau^{k-1} P_a, \tau^l P_b) = \text{Hom}(P_a, \tau^{l-k+1} P_b) \neq 0$ since $\tau^{l-k+1} P_b$ is a sincere preprojective module. If $i > j$

$\text{Ext}^{i - j}(\tau^k P_a[i], \tau^l P_b[j]) = \text{Ext}^1(\tau^k P_a[1], \tau^l P_b) = \text{Hom}(\tau^{l-1} P_a, \tau^k P_b[1]) = \text{Hom}(P_a, \tau^{k-l+1} P_b[1]) \neq 0$ since $\tau^{k-l+1} P_b[1]$ is a sincere preinjective module. Hence $\tau^k P_a$ and $\tau^l P_b$ are silting-incompatible. Exchange the objects if $k - l < -N$. Hence the lemma has been proven.
Now we only need to prove that the minimal transjective degree $L$ of silting objects that can appear in $m$-maximal green sequences has a lower bound.
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Note that due to Lemma 1 there are at least 2 transjective components in any silting object in $D^b(kQ)$. No green mutation within a component or green mutation from a regular component to another one can increase $L$. All other green mutations may increase $L$ by at most $N$. 
Now we only need to prove that the minimal transjective degree $L$ of silting objects that can appear in $m$-maximal green sequences has a lower bound.

Note that due to Lemma 1 there are at least 2 transjective components in any silting object in $D^b(kQ)$. No green mutation within a component or green mutation from a regular component to another one can increase $L$. All other green mutations may increase $L$ by at most $N$.

However there are only $n$ summands of a silting object, $m + 1$ transjective components and $m$ regular components so the amount of mutations that can increase $L$ is at most $2mn$. To reach $\Lambda[m]$ which is of degree 0 $L$ has to always be at least $-2mnN$. 

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Similarly we have an upper bound of maximal transjective degree $H$ of silting objects that can appear in $m$-maximal green sequences, namely $2mnN$. Hence any indecomposable transjective summand that can appear in $m$-maximal green sequences has transjective degree between $-2mnN$ and $2mnN$. 
Similarly we have an upper bound of maximal transjective degree \( H \) of silting objects that can appear in \( m \)-maximal green sequences, namely \( 2mnN \). Hence any indecomposable transjective summand that can appear in \( m \)-maximal green sequences has transjective degree between \( -2mnN \) and \( 2mnN \).

There are finitely many indecomposable transjective summands that can appear in \( m \)-maximal green sequences. Hence there are finitely many indecomposable summands that can appear in \( m \)-maximal green sequences which implies that tame quivers admit finitely many \( m \)-maximal green sequences.
We proved that tame quivers have finitely many $m$-maximal green sequences.
For Further Reading I
