Quasi-abelian hearts of twin cotorsion pairs on triangulated categories

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Set-up & Motivation

- $\mathcal{C} = \text{cluster category (triangulated, Hom-finite, Krull-Schmidt, has Serre duality)}$

- $\Sigma = \text{suspension/shift functor}$
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- $R = \text{rigid, i.e. } \text{Ext}^1_{\mathcal{C}}(R, R) := \text{Hom}_{\mathcal{C}}(R, \Sigma R) = 0$, object of $\mathcal{C}$
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- $\Lambda_R := (\text{End}_{\mathcal{C}} R)^{\text{op}}$
Set-up & Motivation

- $\mathcal{C} =$ cluster category (triangulated, Hom-finite, Krull-Schmidt, has Serre duality)

- $\Sigma =$ suspension/shift functor

- $R =$ rigid, i.e. $\text{Ext}^1_{\mathcal{C}}(R, R) := \text{Hom}_{\mathcal{C}}(R, \Sigma R) = 0$, object of $\mathcal{C}$

- $\Lambda_R := (\text{End}_{\mathcal{C}} R)^{\text{op}}$

**Aim:** to study $\text{mod } \Lambda_R$
\[ C \xrightarrow{\text{Hom}_C(R, -)} \text{mod} \Lambda_R \]

\[ \mathcal{X}_R = \{ X \in C \mid \text{Hom}_C(R, X) = 0 \} \]
where

\[ \mathcal{X}_R = \{ X \in \mathcal{C} \mid \text{Hom}_\mathcal{C}(R, X) = 0 \} \]
A category is \textit{preabelian} if it admits kernels and cokernels.
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![Diagram showing preabelian category](diag.png)
A category is *preabelian* if it admits kernels and cokernels.

\[
\begin{array}{ccccccccc}
\text{Ker } f & \rightarrow & X & \xrightarrow{f} & Y & \rightarrow & \text{Cok } f \\
\downarrow & & \downarrow \circ & & \downarrow \circ & & \downarrow \circ \\
\text{Coim } f & \xrightarrow{\tilde{f}} & \text{Im } f
\end{array}
\]

Such a category is *semi-abelian* if \( \tilde{f} \) is *regular*, i.e. simultaneously monic and epic, for all \( f \).
$\mathcal{C}/\mathcal{X}_R$ is an integral category, i.e. semi-abelian, and PBs of epics are epics and POs of monics are monic.
• \( \mathcal{C}/\mathcal{X}_R \) is an \textit{integral} category, i.e. semi-abelian, and PBs of epics are epics and POs of monics are monic.

• The class \( \mathcal{R} \) of all regular morphisms admits a calculus of fractions. [Rump]
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The class \( \mathcal{R} \) of all regular morphisms admits a calculus of fractions. \[ \text{[Rump]} \]

\[
\begin{array}{ccc}
\mathcal{C} & \xrightarrow{\text{Hom}_\mathcal{C}(R,-)} & \text{mod} \Lambda_R \\
Q \downarrow & \circlearrowleft & \xrightarrow{\cong} \text{Hom}(\mathcal{C}/\mathcal{X}_R)[\mathcal{R}^{-1}](R,-) \\
\mathcal{C}/\mathcal{X}_R & \xrightarrow{L} & (\mathcal{C}/\mathcal{X}_R)[\mathcal{R}^{-1}]
\end{array}
\]
Nakaoka: Cotorsion pairs

Let $S, T$ be nice subcategories of $C$.

**Definition**

A *cotorsion pair* is a pair $(S, T)$ such that:

$\mathcal{C} = S^\perp \Sigma T := \{ X \in \mathcal{C} | \exists \Delta: S \to X \to \Sigma T \to \Sigma S, S \in S, T \in T \}$

$\text{Ext}^1_{\mathcal{C}}(S, T) = 0$

Note: $S = T \perp_1 := \{ X \in \mathcal{C} | \text{Ext}^1_{\mathcal{C}}(X, T) = 0 \}$, and $T = S \perp_1$.

**Example**

The pair $(\text{add } \Sigma R, X R)$ is a cotorsion pair.

Recall $X R = \{ X \in \mathcal{C} | \text{Hom}_{\mathcal{C}}(R, X) = 0 \}$. 
Let $S, T$ be nice subcategories of $C$.

**Definition**

A *cotorsion pair* is a pair $(S, T)$ such that:

- $C = S \ast \Sigma T := \{ X \in C \mid \exists \Delta : S \to X \to \Sigma T \to \Sigma S, \ S \in S, \ T \in T \}$

- $\text{Ext}^1_C(S, T) = 0$

Note: $S = T^1 : = \{ X \in C \mid \text{Ext}^1_C(X, T) = 0 \}$, and $T = S^1$.  

**Example**

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A *cotorsion pair* is a pair $(\mathcal{S}, \mathcal{T})$ such that:

1. $\mathcal{C} = \mathcal{S} \star \Sigma \mathcal{T} := \{X \in \mathcal{C} \mid \exists \Delta : S \to X \to \Sigma T \to \Sigma S, S \in \mathcal{S}, T \in \mathcal{T}\}$
2. $\text{Ext}^1_\mathcal{C}(\mathcal{S}, \mathcal{T}) = 0$
Let $S, T$ be nice subcategories of $C$.

**Definition**

A *cotorsion pair* is a pair $(S, T)$ such that:

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Note: $S = \perp_1 T := \{ X \in C \mid \text{Ext}^1_C(X, T) = 0 \}$, and $T = S \perp_1$. 
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**Definition**

A cotorsion pair is a pair $(S, T)$ such that:

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Note: $S = \perp_{-1} T := \{X \in \mathcal{C} \mid \text{Ext}^1_{\mathcal{C}}(X, T) = 0\}$, and $T = S^\perp_{-1}$.

**Example**

The pair $(\text{add} \Sigma R, \mathcal{X}_R)$ is a cotorsion pair.

Recall $\mathcal{X}_R = \{X \in \mathcal{C} \mid \text{Hom}_{\mathcal{C}}(R, X) = 0\}$.
Let $S, T, U, V$ be nice subcategories of $\mathcal{C}$.

**Definition**

A *twin cotorsion pair* is a pair $((S, T), (U, V))$ such that:

- $(S, T)$ and $(U, V)$ are each cotorsion pairs

Example

$((S, T), (U, V)) = ((\text{add } \Sigma R, X R), (X R, X \perp_1 R))$
Nakaoka: Twin cotorsion pairs

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**Example**

$$((S, T), (U, V)) = ((\text{add } \Sigma R, X_R), (X_R, X_R^\perp_1))$$

**Example**

If $(S, T)$ is a cotorsion pair, then $((S, T), (S, T))$ is a *degenerate* twin cotorsion pair.
Assume from now that \(((S, T), (U, V))\) is a twin cotorsion pair, and define

\[\mathcal{W} := T \cap U,\]
\[\mathcal{C}^- := \Sigma^{-1} S \ast \mathcal{W},\]
\[\mathcal{C}^+ := \mathcal{W} \ast \Sigma V.\]
Nakaoka: the heart

Assume from now that \(((S, T), (U, V))\) is a twin cotorsion pair, and define

\[ \mathcal{W} := T \cap U, \]
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The associated heart is defined to be

\[ \overline{H} := C^- \cap C^+/\mathcal{W} \]
Assume from now that \(((\mathcal{S}, \mathcal{T}), (\mathcal{U}, \mathcal{V}))\) is a twin cotorsion pair, and define
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\mathcal{C}^- := \Sigma^{-1} \mathcal{S} \ast \mathcal{W},
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\mathcal{C}^+ := \mathcal{W} \ast \Sigma \mathcal{V}.
\]

The associated *heart* is defined to be
\[
\overline{\mathcal{H}} := \mathcal{C}^- \cap \mathcal{C}^+ / \mathcal{W}
\]
and is semi-abelian.
Recall: Buan-Marsh show $\mathcal{C}/\mathcal{X}_R$ is integral.
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- If \( ((\mathcal{S}, \mathcal{T}), (\mathcal{U}, \mathcal{V})) = ((\text{add } \Sigma R, \mathcal{X}_R), (\mathcal{X}_R, \mathcal{X}_R^\perp)) \), then

\[ H = \mathcal{C}/\mathcal{X}_R. \]
Nakaoka: recovering Buan & Marsh

Recall: Buan-Marsh show $\mathcal{C}/\mathcal{X}_R$ is integral.

- If $((S, T), (U, V)) = ((\text{add } \Sigma R, \mathcal{X}_R), (\mathcal{X}_R, \mathcal{X}_R^\perp))$, then

$$\overline{H} = \mathcal{C}/\mathcal{X}_R.$$
Recall: Buan-Marsh show $\mathcal{C}/\mathcal{X}_R$ is integral.

- If $((S, \mathcal{T}), (\mathcal{U}, \mathcal{V})) = ((\text{add } \Sigma R, \mathcal{X}_R), (\mathcal{X}_R, \mathcal{X}_R^{\perp 1}))$, then
  \[ \overline{\mathcal{H}} = \mathcal{C}/\mathcal{X}_R. \]

- If $\mathcal{U} \subseteq S \ast \mathcal{T}$ or $\mathcal{T} \subseteq \mathcal{U} \ast \mathcal{V}$, then $\overline{\mathcal{H}}$ is integral.
A quasi-abelian category is a semi-abelian category in which PBs of cokernels are cokernels and POs of kernels are kernels.
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**Theorem (S.)**

Let \(((S, T), (U, V))\) be a twin cotorsion pair on a triangulated category \(\mathcal{C}\). If \(U \subseteq T\) or \(T \subseteq U\), then \(\overline{\mathcal{H}}\) is quasi-abelian.
A quasi-abelian category is a semi-abelian category in which PBs of cokernels are cokernels and POs of kernels are kernels.

**Theorem (S.)**

Let $((S, T), (U, V))$ be a twin cotorsion pair on a triangulated category $\mathcal{C}$. If $U \subseteq T$ or $T \subseteq U$, then $\overline{\mathcal{H}}$ is quasi-abelian.

Setting $((S, T), (U, V)) = ((\text{add } \Sigma R, \mathcal{X}_R), (\mathcal{X}_R, \mathcal{X}_R^{\perp 1}))$, we get

**Corollary**

$\mathcal{C}/\mathcal{X}_R$ is quasi-abelian.
An interesting consequence!

- $\mathcal{C}/\mathcal{X}_R$ is quasi-abelian: a bunch of Auslander-Reiten theory applies! [S.]

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  E.g. irreducible maps must be proper epic or proper monic.

- $\mathcal{C}/\mathcal{X}_R$ is Krull-Schmidt: a bunch of Auslander-Reiten theory applies! [Liu]

  E.g. the AR theory of $\mathcal{C}$ induces the AR theory of $\mathcal{C}/\mathcal{X}_R$. 
Theorem (S.)

Let $\mathcal{A}$ be a Krull-Schmidt quasi-abelian category, and $\xi : X \xrightarrow{f} Y \xrightarrow{g} Z$ an exact sequence in $\mathcal{A}$. The following are equivalent:

1. $\xi$ is an Auslander-Reiten sequence;
2. $\text{End}_\mathcal{A}(X)$ is local and $g$ is right almost split;
3. $\text{End}_\mathcal{A}(Z)$ is local and $f$ is left almost split;
4. $f$ is minimal left almost split; and
5. $g$ is minimal right almost split.