Auslander-Reiten sequences for Gorenstein rings of dimension one

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Abstract

Let R be a complete local Gorenstein ring of dimension one, with maximal ideal $\mathfrak{m}$. We show that a particular endomorphism of $\mathfrak{m}$ produces the Auslander-Reiten sequences of maximal Cohen-Macaulay R-modules. We also adapt results due to Zacharia and others, in the setting of Artin algebras, to the situation of maximal Cohen-Macaulay modules over R.

Throughout, R is a Gorenstein complete local ring of dimension one with maximal ideal $\mathfrak{m}$ and $k = R/\mathfrak{m}$. Let $\mathrm{CM}(R) =$ the category of maximal Cohen-Macaulay R-modules (in our case, $M \in \mathrm{CM}(R) \Leftrightarrow M$ embeds in a free R-module), and $L_p(R) = \{M \in \mathrm{CM}(R) | M_\mathfrak{p}$ is $R_\mathfrak{p}$-free $\forall$ primes $\mathfrak{p} \neq \mathfrak{m}\}$.

If M is a nonfree indecomposable in $L_p(R)$, there exists an (unique up to isomorphism) **Auslander-Reiten (AR) sequence beginning at** M, a short exact sequence $0 \longrightarrow M \xrightarrow{p} E \xrightarrow{q} \mathrm{syz}_R^{-1} M \longrightarrow 0$ in $L_p(R)$ such that any $M \longrightarrow L \in \mathrm{CM}(R)$ which is not a split mono factors through p.

1 Theorem

Let M be a nonfree indecomposable in $L_p(R)$. Then $\mathrm{Ext}^1_R(\mathrm{syz}_R^{-1} M, M) \cong \underline{\mathrm{End}}_R(M) = \mathrm{End}_R(M)/\{$maps factoring through projectives$\}$, and the class $[h]$ of an endomorphism $h\colon M \longrightarrow M$ corresponds to an AR sequence if and only if $[h]$ generates the socle of the local ring $\underline{\mathrm{End}}_R(M)$.

Since M has no free direct summand, there exists no surjection $M \longrightarrow R$, so $M^* = \mathrm{Hom}_R(M, \mathfrak{m})$. Therefore $M \cong \mathrm{Hom}_R(M, \mathfrak{m})^*$ is a module over the ring $\mathrm{End}_R \mathfrak{m}$. **Notation:** If $w \in \mathrm{End}_R \mathfrak{m}$, let $w_M \in \mathrm{End}_R M$ denote corresponding endomorphism of M. For $h \in \mathrm{End}_R M$, let $[h]$ denote the class of h in $\underline{\mathrm{End}}_R M$.

Theorem 1: Assume that there exists a minimal prime $\mathfrak{p}$ of R such that $\mathrm{rank}(M \otimes_R R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p})$ is a unit in R. Then there exists $\gamma \in \mathrm{End}_R \mathfrak{m}$ such that $[\gamma_M]$ generates $\mathrm{soc}(\underline{\mathrm{End}}_R M)$. Such γ is independent of M if M has rank. □

In the case where R is reduced and M has rank, γ can be taken to be any element of $\mathrm{Hom}_R(\mathrm{rad}\, \overline{R}, R) \setminus R$ where $\overline{R}$ denotes the integral closure of R in $Q = R[\text{nonzerodivisors}]^{-1}$. We call such γ a **Frobenius element** (for R).

Example 1 (numerical semigroup ring) Let $R = k[|t^{i_1}, \ldots, t^{i_n}|] \subset k[|t|]$, $\gcd(i_1, ..., i_n) = 1$. Let $v(R) = \mathbb{N}i_1 + \ldots \mathbb{N}i_n$. The *Frobenius number* $F = F(v(R))$ is $\max\{j \in \mathbb{Z} | j \notin v(R)\}$, and t^F is a Frobenius element for R.

Example 2 Let $R = k[|x, y|]/(f)$, where f is an irreducible power series lying in $(x, y)^2$, and $k = \overline{k}$. Assume $f = x^m + y^n +$ mixed terms. Then,

Proposition 1: Assume $\gcd(m, n) = 1$. Then x^{m-1}/y and y^{n-1}/x are Frobenius elements for R.

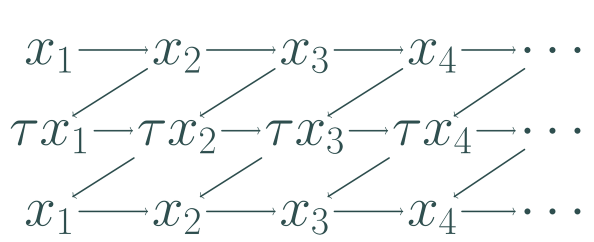
(There are counterexamples when $\gcd(m, n) \neq 1$.)

2 Computing Quiver Components

The stable Auslander-Reiten (AR) quiver of R is the directed graph such that: (1) The vertices are the indecomposable nonfree MCM R-modules N such that $N_\mathfrak{p}$ is $R_\mathfrak{p}$-free for all nonmaximal primes $\mathfrak{p}$, and (2) There is an arrow (i.e. directed edge) $X \to Y$ if and only if there exists an irreducible map $X \longrightarrow Y$.

In some sense this quiver comes from AR sequences, since the irreducible maps can be found from the AR sequences.

Let C be a connected component of the stable AR quiver. The Riedtmann Structure Theorem gives a particular tree T called the tree class of C. If C is periodic, i.e. contains an M such that $\mathrm{syz}^n M \cong M$ for some nonzero integer n, then T must be an infinite Dynkin diagram. From this one obtains a criterion for showing that $T = A_\infty$, which involves the computation of just two AR sequences, and is thus a manageable application of Theorem 1. In this case C is a tube. Below is the picture-definition of a tube "of rank 2"; note that the top and bottom rows should be identified. In our case, $\tau = \mathrm{syz}$.



(The vertices x in a tube of rank 2 satisfy $\tau^2 x = x$.)

3 Adaptation of results of Green, Kerner and Zacharia

Now assume in addition that R is a complete intersection. The following propositions are adapted from results of Green, Kerner and Zacharia in the context of selfinjective Artin algebras.

Proposition 2: If $0 \longrightarrow M \longrightarrow X \longrightarrow N \longrightarrow 0$ is an AR sequence, the number of direct summands of X is bounded by 4.

Definition Given $M, N \in L_p(R)$, an irreducible map $f\colon M \longrightarrow N$ is *syz-perfect* if M and N have no free summands and $\mathrm{syz}^n f$, for $n \geq 0$, are either all monomorphisms or all epimorphisms. If M is a nonfree indecomposable in $L_p(R)$, then M is syz-perfect if every irreducible map $X \longrightarrow M$ and every irreducible map $\mathrm{syz}\, M \longrightarrow Y$ are syz-perfect. M is called *eventually* syz-perfect if $\mathrm{syz}^n M$ is syz-perfect, for some $n \geq 0$.

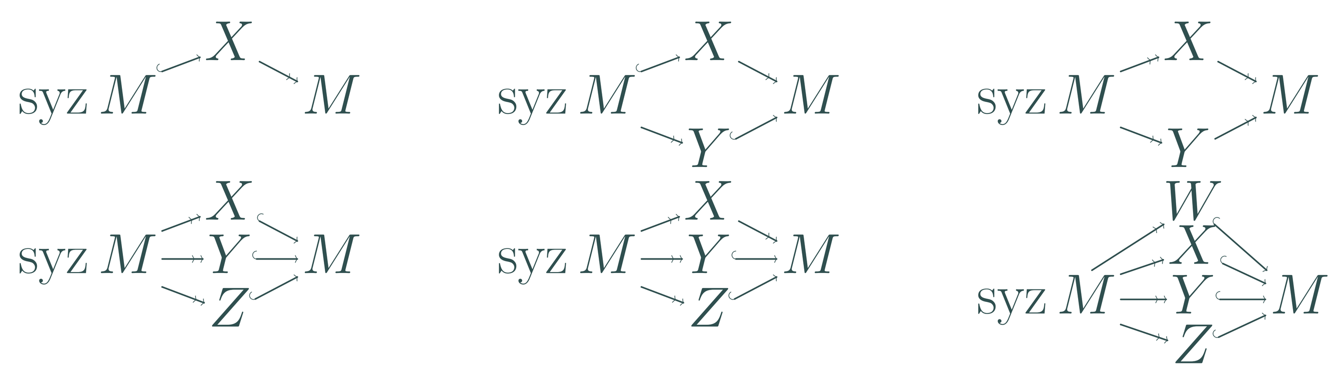
We also have the dual notions of (eventually) cosyz-perfect.

Let $M \in L_p(R)$ be a nonfree, nonperiodic indecomposable, and let C be the component of the stable AR quiver containing M.

Proposition 3: If M is not eventually cosyz-perfect, or not eventually syz-perfect, then C admits an additive function, and R has a periodic ideal.

Proposition 4: Assume every module in C is eventually cosyz-perfect, and $k = \overline{k}$. Then $C \cong \mathbb{Z}\Delta$, where Δ is either a Euclidean diagram of type $\tilde{A}_n$, $\tilde{D}_n$, $\tilde{E}_i$ $(i = 6, 7, 8)$ or a Dynkin diagram of type E_i $(i = 6, 7, 8)$, A_∞, A_∞^∞ or D_∞.

Proposition 5: If every module in C is eventually syz-perfect, and M is syz-perfect, then the AR sequence ending in M has one of the following shapes:



Proposition 5 applies to special cases of the

Huneke-Wiegand Conjecture: Let D be a Gorenstein local domain of dimension one and $M \in \mathrm{CM}(D)$ nonfree. Then $\mathrm{Ext}^1_D(M, M) \neq 0$; equivalently, $\underline{\mathrm{Hom}}_D(\mathrm{syz}_D M, M) \neq 0$.

Namely, the condition $\underline{\mathrm{Hom}}_R(\mathrm{syz}_R M, M) \neq 0$ is easily verified in all but the first two cases in Proposition 5, using the fact that if $f \in \mathrm{Hom}_R(\mathrm{syz}_R M, M)$ factors through a projective then its image lies in $\mathfrak{m}M$.

References

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