Elementary (super) groups

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**Detection Questions**

Let $G$ be some algebraic object so that

$$\text{Rep } G, \quad H^*(G)$$

make sense.

**Question (1)**

How to detect that an element $\xi \in H^*(G)$ is nilpotent?

**Question (2)**

Let $M \in \text{Rep } G$. How to detect projectivity of $M$?

**Question (3)**

$\mathcal{T}(G)$ - tt - category associated to $G$ (stmod $G$, $D^b(G)$, $K(\text{Inj}G)$ ...)

$$\text{supp } M = \emptyset \iff M \cong 0 \text{ in } \mathcal{T}(G)$$
• $G$ - finite group, finite group scheme $H^*(G, k)$.
• $G$ - algebraic group, $H^*(G, A)$
• $G$ - compact Lie group (p-local compact group)
• $G$ - Hopf algebra
  • small quantum group (char 0)
  • restricted enveloping algebra of a p-Lie algebra
  • Lie superalgebra
  • Nichols algebra
• $G$ - finite supergroup scheme

“Other” contexts:
• **Stable Homotopy Theory**: Devinatz - Hopkins - Smith (’88)
• **Commutative Algebra**: $\mathbb{D}^{\text{perf}}(R – \text{mod})$, $\mathbb{D}(R – \text{mod})$, Hopkins (’87), Neeman (’92)
• **Algebraic Geometry**: $\mathbb{D}^{\text{perf}}(\text{coh}(X))$, Thomason (’97)
**Historical framework: finite groups**

**Nilpotence in cohomology:** D. Quillen, B. Venkov, *Cohomology of finite groups and elementary abelian subgroups*, 1972

**Projectivity on elementary abelian subgroups:** L. Chouinard, *Projectivity and relative projectivity over group rings*, 1976

**Projectivity on shifted cyclic subgroup; finite dimensional modules:** E. C. Dade. *Endo-permutation modules over p-groups*, 1978

A $G$-finite group. $k = \overline{\mathbb{F}}_p$.

$\text{Rep } G$ - abelian category with enough projectives ($\text{proj} = \text{inj}$).

- $H^i(G, k) = \text{Ext}^i_G(k, k)$, an abelian group for every $i$.
- $H^*(G, k) = \text{Ext}^*_G(k, k) = \bigoplus \text{Ext}^i_G(k, k)$ - graded commutative algebra;
  $H^*(G, M) = \text{Ext}^*_G(k, M)$ - module over $H^*(G, k)$ via Yoneda product.

**Theorem (Golod ('59), Venkov ('61), Evens('61))**

Let $G$ be a finite group. Then $H^*(G, k)$ is a finitely generated $k$-algebra.
If $M$ is a finite dimensional $G$-module, then $H^*(G, M)$ is a finite module over $H^*(G, k)$. 
$E = (\mathbb{Z}/p)^n$ - an elementary abelian $p$-group of rank $n$.

$$H^*(E, k) = k[Y_1, \ldots, Y_n] \otimes \Lambda^*(s_1, \ldots, s_n), \quad p > 2$$

$E \triangleleft G \leadsto \text{res}_{G, E} : H^*(G, k) \to H^*(E, k)$

**Theorem (Quillen '71, Quillen-Venkov '72)**

A cohomology class $\xi \in H^*(G, k)$ is nilpotent if and only if for every elementary abelian $p$-subgroup $E \triangleleft G$,

$$\text{res}_{G, E}(\xi) \in H^*(E, k)$$

is nilpotent.

We say that nilpotence in cohomology is *detected* on elementary abelian $p$-subgroups.
QUILLEN STRATIFICATION

\[ H^*(E, k) = k[Y_1, \ldots, Y_n] \otimes \Lambda^*(s_1, \ldots, s_n). \]

\[ |E| = \text{Spec } H^*(E, k) = \text{Spec } k[Y_1, \ldots, Y_n] \cong \mathbb{A}^n \]

Theorem (Quillen, '71)

\[ |G| = \text{Spec } H^*(G, k) \text{ is stratified by } |E|, \text{ where } E < G \text{ runs over all elementary abelian } p\text{-subgroups of } G. \]

“Weak form” of Quillen stratification:

\[ |G| = \bigcup_{E < G} \text{res}_{G,E} |E| \]

QUILLEN STRATIFICATION IN GRAPHICS

\[ \text{Spec } H^*(G, \overline{F}_2) \]

for \( G = A_{14} \)

Courtesy of Jared Warner
Spec $H^*(G, \overline{F}_5)$
for $G = GL_4(\mathbb{F}_5)$
Spec $H^*(G, \overline{\mathbb{F}_2})$
for $G = GL_5(\mathbb{F}_2)$
Spec $H^*(G, \overline{F}_2)$ for $G = S_{12}$
Theorem (Chouinard ’76)

Let $G$ be a finite group, and $M$ be a $G$-module. Then $M$ is projective if and only for any elementary abelian $p$-subgroup $E$ of $G$, $M_{\downarrow E}$ is projective.

“Projectivity is detected on elementary abelian $p$-subgroups”.
What about elementary abelian $p$-subgroups?
What about elementary abelian $p$-subgroups?
Let $E = (\mathbb{Z}/p)^n$, $(\sigma_1, \sigma_2, \ldots, \sigma_n)$ be generators of $E$. Then

$$kE \cong \frac{k[\sigma_1, \sigma_2, \ldots, \sigma_n]}{(\sigma_i^p - 1)} \cong \frac{k[x_1, \ldots, x_n]}{(x_1^p, \ldots, x_n^p)}.$$  

where $x_i = \sigma_i - 1$.

$$\lambda = (\lambda_1, \ldots, \lambda_n) \in k^n \implies X_\lambda = \lambda_1 x_1 + \cdots + \lambda_n x_n \in kE.$$  

Freshman calculus rule: $X_\lambda^p = 0$, $(X_\lambda + 1)^p = 1$.
Hence, $\langle X_\lambda + 1 \rangle \cong \mathbb{Z}/p$ is a shifted cyclic subgroup of $kE$.

**Theorem (Dade’78)**

Let $E$ be an elementary abelian $p$-group, and $M$ be a finite dimensional $E$-module. Then $M$ is projective if and only if for any $\lambda \in k^n \setminus \{0\}$, 
$M \downarrow_{\langle X_\lambda + 1 \rangle}$ is projetive (free).
APPLICATIONS

- Support varieties for $G$-modules (Alperin-Evans, Carlson, Avrunin-Scott, ...)

- Classification of thick tensor ideals in $\text{stmod } G$; localizing tensor ideals in $\text{Stmod } G$ (Benson-Carlson-Rickard’97; Benson-Iyengar-Krause’11)

- Computation of Balmer spectrum of $\text{stmod } G$. 

Finite group schemes

An affine group scheme over $k$ is a representable functor

$$G : \text{comm } k - \text{alg} \rightarrow \text{groups}$$

$R$ - commutative $k$-algebra. $\leadsto G(R) = \text{Hom}_{k-\text{alg}}(k[G], R)$.

$k[G]$ is a commutative Hopf algebra.

An affine group scheme is finite if $\dim_k k[G] < \infty$.

$$\left\{ \begin{array}{c} \text{finite group schemes} \\ G \end{array} \right\} \sim \left\{ \begin{array}{c} \text{finite dimensional} \\ \text{commutative} \\ \text{Hopf algebras} \\ k[G] \end{array} \right\}$$
$G$ - a finite group scheme.

$kG := k[G]^\vee = \text{Hom}_k(k[G], k)$, the group algebra of $G$, a finite-dimensional cocommutative Hopf algebra

\[ \begin{cases} 
\text{finite group schemes } G \\
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\[ \text{Rep}_k G \sim k[G]\text{-comodules} \sim kG\text{-modules} \]
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\end{align*}
\]

Abuse of language: G-modules

$\text{Rep } G = \text{Mod } G$ - abelian category with enough projectives (proj= inj)

$H^*(G, k) = H^*(kG, k)$ - graded commutative algebra.
EXAMPLES

- **Finite groups.** $kG$ is a finite dimensional cocommutative Hopf algebra, generated by group like elements.

- **Restricted Lie algebras.**
  Let $\mathcal{G}$ be an algebraic group ($GL_n, SL_n, Sp_{2n}, SO_n$).
  Then $\frak{g} = \text{Lie } \mathcal{G}$ is a *restricted Lie algebra*. It has the $p$-restriction map (or $p^\text{th}$-power map)

  $$[p] : \frak{g} \to \frak{g}$$

  a semi-linear map satisfying some natural axioms.
  For example, for $\frak{g} = gl_n$, $A^{[p]} = A^p$

  $$u(\frak{g}) = U(\frak{g})/\langle x^p - x^{[p]}, x \in \frak{g} \rangle$$

  restricted enveloping algebra (f.d. cocommutative Hopf algebra).
**Examples**

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  \[
  \text{Representations of } \mathfrak{g} \sim u(\mathfrak{g})\text{-modules}
  \]
• **Frobenius kernels.** $F : \mathcal{G} \to \mathcal{G}$ - Frobenius map;

\[
\mathcal{G}_r = \ker \{F^r : \mathcal{G} \to \mathcal{G}\}
\]

(connected) finite group scheme.

• **Frobenius kernels of the Additive group $\mathbb{G}_a$.**

$\mathbb{G}_a(R) := R^+$. \\
$k[\mathbb{G}_a] = k[T], \Delta(T) = T \otimes 1 + 1 \otimes T$.

\[
F : \mathbb{G}_a \xrightarrow{a \mapsto a^p} \mathbb{G}_a
\]

$\mathbb{G}_{a(1)}(R) = \ker F(R) = \{a \in R \mid a^p = 0\}$

$\mathbb{G}_{a(r)}(R) = \ker F^r(R) = \{a \in R \mid a^{p^r} = 0\}$

\[
k[\mathbb{G}_{a(r)}] \cong k[T]/T^{p^r}; \quad \Delta(T) = T \otimes 1 + 1 \otimes T
\]

$\kappa \mathbb{G}_{a(r)} \cong k[s_1, \ldots, s_n]/(s_1^p, \ldots, s_n^p)$

Coproduct in $k\mathbb{G}_{a(r)}$ is given by "Witt polynomials".
FINITE GENERATION OF COHOMOLOGY

Theorem (Friedlander-Suslin, ’97)

Let $A = kG$ be a finite dimensional cocommutative Hopf algebra over a field $k$. Then $H^*(A, k)$ is a finitely generated $k$-algebra.

If $M$ is a finite dimensional $A$-module, then $H^*(A, M)$ is a finite module over $H^*(A, k)$. 


Nilpotence and projectivity, infinite dimensional modules, *unipotent finite groups schemes*. C. Bendel, *Cohomology and projectivity of modules for finite group schemes*, 2001


Nilpotence, *all finite groups schemes*. A. Suslin, *Detection theorem for finite groups schemes*, 2006


D. Benson, S. Iyengar, H. Krause, J. Pevtsova, *Stratification of module categories for finite groups schemes*, 2018
Definition

An elementary group scheme is a finite group scheme isomorphic to $\mathbb{G}_a(r) \times (\mathbb{Z}/p)^\times n$.

The group algebra is commutative and cocommutative; as an algebra it looks like $kE$ for $E$ an elementary abelian $p$-group. As a coalgebra it is (way) more complicated but still very explicit.

Definition

A $\pi$-point $\alpha$ of a finite group scheme $G$ defined over field extension $K/k$ is a flat map of algebras

$$K[t]/t^p \xrightarrow{\alpha} KG$$

which factors through some elementary subgroup scheme $\mathcal{E} \subset G_K$. 
Theorem (Suslin’06)

Let $G$ be a finite groups scheme. A class $\zeta \in H^* (G, k)$ is nilpotent if and only if

$$\text{res}_{G_K, E} (\zeta_K) \in H^* (E, K)$$

is nilpotent for any field extension $K/k$ and any elementary subgroup scheme $E < G_K$.

Theorem (Benson-Iyengar-Krause-P’18)

Let $G$ be a finite group scheme, and $M$ be a $G$-module. Then $M$ is projective if and only if for every field extension $K/k$ and any $\pi$-point $\alpha : K[t]/t^\pi \to KG$, the $K[t]/t^\pi$-module $\alpha^* (M_K)$ is projective (free).

Generalization of Dade + Chouinard in two directions: to all finite group schemes ($\sim$ finite dimensional cocommutative Hopf algebras), and to infinite dimensional modules.

Finite generation + detection “$\Rightarrow$” Theory of supports in Stmod $G$
FINITE SUPERGROUP SCHEMES

char $k = p > 2$, $\bar{k} = k$ (perfect is enough)
$\mathbb{Z}/2$-graded vector spaces, $\mathbb{Z}/2$-graded Hopf algebras
$A = A_{ev} \oplus A_{odd}$

Graded commutative: $a \cdot b = (-1)^{|a||b|} b \cdot a$

Graded cocommutative: $T \circ \Delta = \Delta$, where $\Delta$ is the coproduct,
$T : V \otimes W \to W \otimes V$;
$T(v \otimes w) = (-1)^{|v||w|} w \otimes v$.

\[
\begin{cases}
\text{finite supergroup schemes } G \\
\end{cases}
\sim
\begin{cases}
\text{finite dimensional } \\
\mathbb{Z}/2\text{-graded cocommutative} \\
\text{Hopf algebras } A = kG \\
\end{cases}
\]
**Examples**

- Finite group schemes (∼ finite dimensional cocommutative Hopf algebras): $G = G_{ev}$.
- Restricted Lie superalgebras $\mapsto$ restricted enveloping algebras $\mapsto$ f.d. graded cocommutative Hopf algebras.

**Definition**

$G^-_a$ is a finite supergroup scheme with coordinate algebra

$$\Lambda^*(v) \simeq k[v]/v^2, \ |v| = 1, \ \Delta(v) = v \otimes 1 + 1 \otimes v$$

- $G^-_a$ is self-dual with group algebra $kG^-_a = k[\sigma]/\sigma^2, \ |\sigma| = 1$.
- Exterior algebras $\Lambda^*(V)$, corresponding to $G^-_a \times \ldots \times G^-_a$
- Finite dimensional sub Hopf algebras of the $\text{mod } p$ Steenrod algebra ($\mathbb{Z}$-graded).
**Cohomology**

\[
\text{Rep } G = \text{Mod } kG - \text{ super } k\text{-vector spaces with linear } kG\text{-action.}
\]

Cohomology \( H^{*,*}(G, k) = H^{*,*}(kG, k) \)
**Cohomology**

\[ \text{Rep } G = \text{Mod } kG \text{ -- super } k\text{-vector spaces with linear } kG\text{-action.} \]

Cohomology \( H^{*,*}(G, k) = H^{*,*}(kG, k) \) - cohomological degree
Cohomology

\[ \text{Rep } G = \text{Mod } kG – \text{super } k\text{-vector spaces with linear } kG\text{-action.} \]

Cohomology \( H^{*,*}(G, k) = H^{*,*}(kG, k) \) - internal degree
**Cohomology**

Rep $G = \text{Mod} \ kG$ – super $k$-vector spaces with linear $kG$-action.

Cohomology $H^{\ast,\ast}(G, k) = H^{\ast,\ast}(kG, k)$

**Theorem (Drupieski’16)**

Let $G$ be a finite supergroup scheme. Then $H^{\ast,\ast}(G, k)$ is a finitely generated $k$-algebra.

For detection, we need “elementary supergroups”.
**Witt Vectors**

\[ W : \text{comm } k - \text{algebras} \rightarrow \text{groups} \]

affine group scheme of **additive Witt vectors**.

\[ W(R) = \{(a_0, a_1, \ldots) | a_i \in R\} \]

\[ (a_0, a_1, \ldots) + (b_0, b_1, \ldots) = (S_0(a_0, b_0), S_1(a_0, a_1, b_0, b_1), \ldots), \]

\( S_i \)-structure polynomials for the additive Witt vectors.

For example, \( S_0 = a_0 + b_0, S_1 = a_1 + b_1 + \frac{(a_0+b_0)^p-a_0^p-b_0^p}{p} \).

**\( W_m \)** - the group scheme of Witt vectors of length **\( m \)**

**\( W_{m,n} := W_m(n) \)** - the **\( n \)**th Frobenius kernel of **\( W_m \)**

- a finite connected commutative unipotent group scheme.

**Examples:**

- **\( W_1 \cong \mathbb{G}_a \), **\( W_{1,n} \cong \mathbb{G}_a(n) \)
- **\( W_{m,1} \cong \mathbb{G}_a^\vee(m) \)** (Cartier dual)
$W_{2,2}(R) = \{(a_0, a_1) | a_0, a_1 \in R\}; \quad kW_{2,2} \cong k[s_0, s_1]/(s_0^{p^2}, s_1^{p^2})$

\[
\Delta(s_0) = S_0(s_0 \otimes 1, 1 \otimes s_0) = s_0 \otimes 1 + 1 \otimes s_0
\]

\[
\Delta(s_1) = S_1(s_0 \otimes 1, s_1 \otimes 1, 1 \otimes s_0, 1 \otimes s_1) = s_1 \otimes 1 + 1 \otimes s_1 + \frac{(s_0 \otimes 1 + 1 \otimes s_0)^p - (s_0 \otimes 1)^p - (1 \otimes s_0)^p}{p}
\]

The simple quotients are $\mathbb{G}_{a(1)}$. 
$W_{m,n}$
$E_{m,n}$
$E_{m,n}$

$E_{m,n}^-$

$G_a^-$

$E_{m,n}$
WITT ELEMENTARY SUPERGROUP SCHEMES

(Super) technical part: Witt elementary supergroup schemes

\[ \mathbb{E}_{m,n}^- = \frac{k[s_1, \ldots, s_{n-1}, s_n, \sigma]}{(s_1^p, \ldots, s_{n-1}^p, s_n^m, \sigma^2 - s_n^p)} \]

\(s_1, \ldots, s_n\) are even; \(\sigma\) is odd.

\[
\Delta(s_i) = S_{i-1}(s_1 \otimes 1, \ldots, s_i \otimes 1, 1 \otimes s_1, \ldots, 1 \otimes s_i) \quad (i \geq 1)
\]

\[
\Delta(\sigma) = \sigma \otimes 1 + 1 \otimes \sigma
\]

where the \(S_i\) are the structure polynomials for the Witt vectors.
Definition

A finite supergroup scheme is elementary if it’s isomorphic to a quotient of $\mathbb{E}_{m,n}^- \times (\mathbb{Z}/p)^s$.

Remark: These quotients can be explicitly classified using the theory of Diedonné modules.

Theorem (Classification)

An elementary supergroup scheme is isomorphic to one of the following:

(i) $G_{a(n)} \times (\mathbb{Z}/p)^s$,
(ii) $G_{a(n)} \times G_{a}^- \times (\mathbb{Z}/p)^s$,
(iii) $E_{m,n}^- \times (\mathbb{Z}/p)^s$,
(iv) $E_{m,n,\mu}^- \times (\mathbb{Z}/p)^s$. 


\[ k\mathbb{E}_{m,n,\mu} = \frac{k[s_1, \ldots, s_{n-1}, s_n, \sigma]}{(s_1^p, \ldots, s_{n-1}^p, s_n^{p+1}, \sigma^2 - s_n^p)} \]

\[ \Delta(s_i) = S_i(\mu s_n^p \otimes 1, s_1 \otimes 1, \ldots, s_i \otimes 1, 1 \otimes \mu s_n^p, 1 \otimes s_1, \ldots, 1 \otimes s_i) \]

\[ \Delta(\sigma) = \sigma \otimes 1 + 1 \otimes \sigma \]
\[ k \mathbb{E}_{m,n,\mu}^- = \frac{k[s_1, \ldots, s_{n-1}, s_n, \sigma]}{(s_1^p, \ldots, s_{n-1}^p, s_n^{p+1}, \sigma^2 - s_n)} \]

\[ \Delta(s_i) = S_i(\mu s_n^m \otimes 1, s_1 \otimes 1, \ldots, s_i \otimes 1, 1 \otimes \mu s_n^m, 1 \otimes s_1, \ldots, 1 \otimes s_i) \]

\[ \Delta(\sigma) = \sigma \otimes 1 + 1 \otimes \sigma \]
**Detection Theorem**

Theorem (Benson-Iyengar-Krause-P’18)

Suppose that $G$ is a finite unipotent supergroup scheme. Then

(i) Nilpotence of elements of $H^*,*(-G,k)$ and

(ii) Projectivity of $G$-modules

are detected upon restriction to sub supergroup schemes isomorphic to a quotient of some $\mathbb{E}_{m,n}^- \times (\mathbb{Z}/p)^s$ (after field extension).
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THANK YOU