Gorenstein Projective Modules for the Working Algebraist

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Overview

1 Background
- Definition and Properties
- Applications

2 The explicit construction of Gorenstein projective modules
- Upper Triangular Matrix Rings
- Path Algebras of Acyclic Quivers
- Tensor Products of algebras
Let $R$ be a ring. A module $M$ is **Gorenstein projective**, if there exists a complete projective resolution

$$P^\bullet = \cdots \to P^{-1} \to P^0 \xrightarrow{d^0} P^1 \to \cdots$$

such that $M \cong \text{Ker } d^0$. 

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Avramov, Buchweitz, Martsinkovsky and Reiten proved that a finitely generated module $M$ over Noetherian ring $R$ is Gorenstein projective if and only if $\text{G-dim}_R M = 0$. 
Theorem (Henrik Holm 2004)

Let $R$ be a non-trivial associative ring. Then $\mathcal{GP}(R)$ is projectively resolving. That is to say, $\mathcal{GP}(R)$ contains the projective modules and is closed under extensions, direct summands, kernels of surjections.

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Let $R$ be an Artin Gorenstein ring, then $\mathcal{GP}(R)$ is a Frobenius category whose projective-injective objects are exactly all the projective $R$-modules.
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**Theorem**

If $R$ is a Gorenstein ring, then $\mathcal{GP}(R)$ is contravariantly finite [Enochs and Jenda 1995], thus it is functorially finite, and hence $\mathcal{GP}(R)$ has AR-seqs [Auslander and Smalø 1980].
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Applications

- Singularity theory: \( \mathcal{GP}(R) \cong D_{sg}(R) \) as triangular categories,
  Buchweitz: when \( R \) is Gorenstein Noetherian ring;
  Happel: when \( R \) is Gorenstein algebra.
  Ringel and Pu Zhang: \( \mathcal{GP}(kQ \otimes_k k[x]/(X^2)) \cong D^b(kQ)/[1] \).
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- Tate cohomology theory: $\hat{\text{Ext}}_R^n(M, N) = H^n\text{Hom}_R(T, N)$ where $T$ is a complete projective resolution in a complete resolution $T \xrightarrow{\nu} P \xrightarrow{\pi} M$ with $\nu_n$ bijection when $n \gg 0$. [Avramov and Martsinkovsky]

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- the invariant subspaces of nilpotent operators:
  Ringel and Schmidmeier: $\{(V, U, T) \mid T : V \to V, T^6 = 0, U \subset V, T(U) \subset U\} = \mathcal{GP}(k[T]/(T^6) \otimes_k k(\bullet \to \bullet))$;
  Kussin, Lenzing and Meltzer showed a surprising link between singularity theory and the invariant subspace problem of nilpotent operators.

...
Let $A$ and $B$ be rings, $M$ an $A-B$–bimodule, and $T := \begin{pmatrix} A & AM_B \\ 0 & B \end{pmatrix}$. Assume that $T$ is an Artin algebra and consider finitely generated $T$–modules. A $T$–module can be identified with a triple $\left( X \atop Y \right)_\phi$, where $X \in A\text{-mod}$, $Y \in B\text{-mod}$, and $\phi : M \otimes_B Y \to X$ is an $A$–map. $\mathcal{G}_p(T)$ is the category of finitely generated Gorenstein proj. $T$–modules.
The explicit construction of Gorenstein projective modules

Let $A$ and $B$ be rings, $M$ an $A-B$-bimodule, and $T := \begin{pmatrix} A & A^M_B \\ 0 & B \end{pmatrix}$.

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$Gp(T)$ is the category of finitely generated Gorenstein proj. $T$-modules.

**Theorem 2.1 (P. Zhang 2013)**

Let $A$ and $B$ be algebras and $M$ a $A-B$-bimodule with $\text{pdim}_A M < \infty$, $\text{pdim}M_B < \infty$, $T := \begin{pmatrix} A & A^M_B \\ 0 & B \end{pmatrix}$. Then $(X \ Y)_{\phi} \in Gp(T)$ if and only if $\phi : M \otimes_B Y \to X$ is an injective $A$-map, $\text{Coker} \phi \in Gp(A)$ and $Y \in Gp(B)$.
Let $Q = (Q_0, Q_1, s, e)$ be a finite acyclic quiver, $k$ a field, $A$ a f. d. $k$-algebra. Label the vertices as $1, 2, \cdots, n$ such that for each arrow $\alpha$, $s(\alpha) > e(\alpha)$. Then $A \otimes_k kQ$ is equivalent to an upper triangular algebra.
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**Theorem 2.2 (joint with P. Zhang 2013)**

Let $Q$ be a finite acyclic quiver, and $A$ a finite dimensional algebra over a field $k$. Let $X = (X_i, X_\alpha)$ be a representation of $Q$ over $A$. Then $X$ is Gorenstein projective if and only if $X$ is separated monic, and $\forall i \in Q_0$, $X_i \in Gp(A)$, $X_i / (\sum_{\alpha \in Q_1, e(\alpha)=i} \text{Im} X_\alpha) \in Gp(A)$. 

**Definition 2.3 separated monic representation**

A representation $X = (X_i, X_\alpha)$ of $Q$ over $A$ is separated monic, if for each $i \in Q_0$, the $A$-map $\bigoplus_{\alpha \in Q_1, e(\alpha)=i} e(\alpha) \rightarrow X_i$ is injective.
Let $Q = (Q_0, Q_1, s, e)$ be a finite acyclic quiver, $k$ a field, $A$ a f. d. $k$-algebra. Label the vertices as $1, 2, \cdots, n$ such that for each arrow $\alpha, s(\alpha) > e(\alpha)$. Then $A \otimes_k kQ$ is equivalent to an upper triangular algebra.

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**Definition 2.3 separated monic representation**

A representation $X = (X_i, X_\alpha)$ of $Q$ over $A$ is **separated monic**, if for each $i \in Q_0$, the $A$-map $\bigoplus_{\alpha \in Q_1, e(\alpha) = i} X_{s(\alpha)} \xrightarrow{(X_\alpha)} X_i$ is injective.
In fact, let \( \Lambda = A \otimes_k kQ \), \( D = \text{Hom}_k(-, k) \), \( S_i \) is a simple left \( kQ \)-module,

\[
0 \to \bigoplus_{\alpha \in Q_1} e_{s(\alpha)} kQ \xrightarrow{(\alpha, \cdot)} e_i kQ \to D(S_i) \to 0, \text{ exact}
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In fact, let $\Lambda = A \otimes_k kQ$, $D = \text{Hom}_k(-, k)$, $S_i$ is a simple left $kQ$-module,

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$$0 \to \bigoplus_{\alpha \in Q_1} A \otimes e_s(\alpha) kQ \overset{(1 \otimes \alpha)}{\to} A \otimes e_i kQ \to A \otimes D(S_i) \to 0, \text{ exact}$$
In fact, let $\Lambda = A \otimes_k kQ$, $D = \text{Hom}_k(-, k)$, $S_i$ is a simple left $kQ$-module,

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$$0 \to \bigoplus_{\alpha \in Q_1 \atop e(\alpha) = i} (1 \otimes e_{s(\alpha)}) \Lambda \xrightarrow{(1 \otimes \alpha \cdot)} (1 \otimes e_i) \Lambda \to A \otimes D(S_i) \to 0, \text{ exact}$$
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$$0 \rightarrow \bigoplus_{\alpha \in Q_1 \atop e(\alpha) = i} X_s(\alpha) \xrightarrow{(X_\alpha)} X_i \rightarrow (A \otimes D(S_i)) \otimes_\Lambda X \rightarrow 0 \ (\ast)$$
\((\ast)\) is exact if and only if \(\bigoplus_{\alpha \in Q_1} X_{s(\alpha)} \xrightarrow{(X_\alpha)} X_i\) is injective if and only if \(\text{Tor}^\Lambda_i (A \otimes_k D(S_i), X) = 0\) for all \(i \geq 1\) and all simple left \(kQ\)-modules \(S_i\).
\((\ast)\) is exact if and only if \(\bigoplus_{\alpha \in Q_1} X_{s(\alpha)} \xrightarrow{(X_\alpha)} X_i\) is injective if and only if

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\text{Tor}_{i}^\Lambda(A \otimes_k D(S_i), X) = 0 \quad \text{for all } i \geq 1 \text{ and all simple left } kQ\text{-modules } S_i.
\]

**Definition 2.4 (Generalized) separated monic representation**

Let \(k\) be a field, \(A\) and \(B\) finite dimensional \(k\)-algebras, \(\Lambda := A \otimes_k B\). A left \(\Lambda\)-module \(X\) is called a **(generalized) separated monic representation** of \(B\) over \(A\), if

\[
\text{Tor}_{i}^\Lambda(A \otimes_k D(S), X) = 0
\]

for all \(i \geq 1\) and all simple left \(B\)-modules \(S\).

\(\text{sm}{\text{on}}(B, A)\): the category of separated monic representation of \(B\) over \(A\).
Define

\[ \text{sm} \text{on}(B, Gp(A)) := \{ X \in \text{sm} \text{on}(B, A) \mid (A \otimes_k V) \otimes \Lambda X \in Gp(A), \forall V_B \}. \]
Define

\[ \text{sm}(B, \mathcal{G}p(A)) := \{ X \in \text{sm}(B, A) \mid (A \otimes_k V) \otimes \Lambda X \in \mathcal{G}p(A), \ \forall \ V_B \}. \]

**Proposition 2.5**

Let \( A \) and \( B \) be f. d. \( k \)-algebras. Then \( \text{sm}(B, \mathcal{G}p(A)) \subset \mathcal{G}p(\Lambda) \).
Define

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Let \( A \) and \( B \) be f. d. \( k \)-algebras. Then \( \text{sm} \! \text{on}(B, \mathcal{G}p(A)) \subset \mathcal{G}p(\Lambda) \).

**Question:** When does \( \mathcal{G}p(\Lambda) \) coincide with \( \text{sm} \! \text{on}(B, \mathcal{G}p(A)) \)?

**Theorem 2.6 (joint with W. Hu, B. Xiong and G. Zhou 2018)**

Suppose that \( B \) is Gorenstein. Then \( \text{sm} \! \text{on}(B, \mathcal{G}p(A)) = \mathcal{G}p(\Lambda) \) if and only if \( \text{gl} \! \text{.dim}(B) < \infty \).

Suppose that \( A \) is Gorenstein. Then \( \text{sm} \! \text{on}(B, \mathcal{G}p(A)) = \mathcal{G}p(\Lambda) \) if and only if \( B \) is CM-free.
Define

\[ \text{smom}(B, \mathcal{G}p(A)) := \{ X \in \text{smom}(B, A) \mid (A \otimes_k V) \otimes \Lambda X \in \mathcal{G}p(A), \ \forall \ V_B \}. \]

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Define

\[ \text{smon}(B, \mathcal{G}p(A)) := \{ X \in \text{smon}(B, A) \mid (A \otimes_k V) \otimes \Lambda X \in \mathcal{G}p(A), \ \forall \ V_B \}. \]

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Via filtration categories

\[ \mathcal{G} p(A) \otimes \mathcal{G} p(B) := \{ X \otimes_k Y \in A \otimes_k B \mod | X \in \mathcal{G} p(A), \ Y \in \mathcal{G} p(B) \} \]

\[ \tilde{\text{filt}}(\mathcal{G} p(A) \otimes \mathcal{G} p(B)) \subset \mathcal{G} p(A \otimes_k B) \]
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Let \( A \) and \( B \) be Gorenstein algebras. Assume that \( k \) is a splitting field for \( A \) or \( B \). Then \( \mathcal{G}p(A \otimes_k B) = \widetilde{\text{filt}}(\mathcal{G}p(A) \otimes \mathcal{G}p(B)) \).
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- Let \( A \) and \( B \) be Gorenstein algebras. Assume that \( k \) is a splitting field for \( A \) or \( B \). Then \( \mathcal{G}p(A \otimes_k B) = \tilde{\text{filt}}(\mathcal{G}p(A) \otimes \mathcal{G}p(B)) \).
- Let \( A \) be an algebra, and let \( B \) be a upper triangular algebra such that \( k \) is a splitting field for \( B \). Then \( \mathcal{G}p(A \otimes_k B) = \tilde{\text{filt}}(\mathcal{G}p(A) \otimes \mathcal{G}p(B)) \).


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Thank You!