The postr of exact structures and Gabriel-Roiter measure
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Reduction of exact structures

We study exact categories in the sense of Quillen [Q]. Consider a class $E$ of kernel-cokernel pairs $(i, d)$ on an additive category $A$. The morphisms $(i, d) \in E$ are referred to as admissible monics and admissible epics. Then $E$ is called an exact structure on $A$ if it is closed under isomorphisms and satisfies the following axioms:

(0) For all objects $A \in Ob(A)$ the identity $1_A$ is an admissible monic and an admissible epic.

(A1) The class of admissible monics is closed under composition, likewise for admissible epics.

(A2) The push-out of an admissible monic along an arbitrary morphism exists and yields an admissible monic; the pull-back of an admissible epic along an arbitrary morphism exists and yields an admissible epic.

Definition: A reduction of an exact structure $(A, E)$ is the choice of an exact structure $E' \subseteq E$ giving rise to a new exact category $(A, E')$. Our main result to reduce exact structures is exactly the following function:

Definition: Let $(A, E_s)$ and $(B, E)$ be exact categories and $F : A \to B$ an exact functor, that is, the $(F_i, F_d)$ of each exact pair $(i, d) \in E_s$ is exact in $B$. We define the following structure $E' = \{ (\epsilon ; 0 : A \to B \to C \to 0 \mid F(\epsilon) \text{ is split exact in } B \} \subseteq E_A$ where $\epsilon$ are short exact sequences in $E_A$.

\[ \text{Lemma (Reduction of length)} \]
If $E$ and $E'$ are exact structures on $A$, such that $E' \subseteq E$, then $l_2(X) \leq l_1(X)$ for all objects $X$ of $A$.

Example 2. An $E'$-simple object in an exact category $(A, E')$ is ind-composable, since the canonical inclusion $X_0 = X_1 \oplus X_2$ is admissible in every exact structure $E_1$ [Bü]. Conversely, when $E$ is the split exact structure $E_{\text{split}}$, then every ind-composable object is $E_{\text{split}}$-simple and $l_{E_{\text{split}}}(X) = 1$. By taking $X \in \text{ind}_A$ where $A = \text{rep}(Q)$ and $Q = A_3$ as in example 3, we have for $X_{\text{split}} \subseteq E_{\text{split}} \subseteq E_A$ that $l_{E_{\text{split}}}(X) = 1 < l_{E_{\text{split}}}(X) = 2 < l_{E}(X) = 3$.

Let $(A, E)$ be an $E'$-finite essentially small exact category. A morphism of exact structures $\phi : (A, E) \to (P, \Theta)$ is called a Gabriel-Roiter measure on the exact category $(A, E)$ if it verifies the following axioms:

(1) $\phi$ is a measure.

(2) $\phi$ is $\Theta$-ind-composable and $\phi(X) \subseteq \Theta(X)$ for all $X \in \text{ind}_A$.

Theorem: There exist a Gabriel-Roiter measure for $\text{ind}_A$.

Proof: For a fixed object $X \in \text{ind}_A$, we consider the filtrations ending by $X$ and we consider all vectors $\phi(X)$.

One can define

\[ \begin{align*}
\phi(X) &= \text{max}\{\mu(X) \mid X \in \text{ind}_A\} \\
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\end{align*} \]

where $\Theta(N)$ is the set of all vectors of natural numbers and $\mu$ is the inverse lexicographic order, so $(\Theta(N), \leq)$ forms a totally ordered set. One can verify that $\phi$ satisfies the three axioms of a Gabriel-Roiter measure.

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References

