The poset of exact structures and Gabriel-Roiter measure

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Reduction of exact structures

We study exact categories in the sense of Quillen [Qu]: Consider a class $\mathcal{E}$ of kernel-cokernel pairs (i, d) on an additive category $\mathcal{A}$. The morphisms (i, d) in $\mathcal{E}$ are referred to as admissible monics and admissible epics. Then $\mathcal{E}$ is called an *exact structure* on $\mathcal{A}$ if it is closed under isomorphisms and satisfies the following axioms:

(A0) For all objets $A \in Obj\mathcal{A}$ the identity 1_A is an admissible monic and an admissible epic.

(A1) The class of admissible monics is closed under composition, likewise for admissible epics.

(A2) The push-out of an admissible monic along an arbitrary morphism exists and yields an admissible monic; the pull-back of an admissible epic along an arbitrary morphism exists and yields an admissible epic.

Definition: A reduction of an exact category $(\mathcal{A}, \mathcal{E})$ is the choice of an exact structure $\mathcal{E}' \subseteq \mathcal{E}$ giving rise to a new exact category $(\mathcal{A}, \mathcal{E}')$.

Our main method to reduce exact structures is using exact functors:
Definition: Let $(\mathcal{A}, \mathcal{E}_\mathcal{A})$ and $(\mathcal{B}, \mathcal{E}_\mathcal{B})$ be exact categories and $F : \mathcal{A} \to \mathcal{B}$ an exact functor, that is, the image (Fi, Fd) of each exact pair (i, d) in $\mathcal{E}_\mathcal{A}$ is exact in $\mathcal{B}$. We define the following structure

$$\mathcal{E}_F = \{\xi : 0 \to A \to B \to C \to 0 \mid F(\xi) \text{ is split exact in } \mathcal{B}\} \subset \mathcal{E}_\mathcal{A}$$

where ξ are short exact sequences in $\mathcal{E}_\mathcal{A}$.

Proposition

$(\mathcal{A}, \mathcal{E}_F)$ is an exact category.

Example 1. Let Q be the quiver

$$1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3$$

and $\mathcal{A} = \text{rep}\, Q$ the category of finite-dimensional representations of Q, endowed with the abelian exact structure $\mathcal{E}_{ab}$ of all short exact sequences.

- Consider the full subquiver $Q' = 1 \xrightarrow{\alpha} 2$ of Q, and let F be the restriction functor

  $$F = \text{Res}_{Q'} : \text{rep}\, Q \to \text{rep}\, Q'.$$

  Then the exact structure $\mathcal{E}_F = \mathcal{E}_\alpha$ is formed by all short exact sequences of rep Q that split when restricted to the subquiver Q'. Moreover $\mathcal{E}_\alpha$ coincides with the exact structure $\mathcal{E}_{2,3,5}$ from example 3.

- Now consider the quiver $Q' = 1 \xrightarrow{\gamma} 3$, and let $G : \text{rep}\, Q \to \text{rep}\, Q'$ be the contraction functor given by

  $$G(V_1 \xrightarrow{V_\alpha} V_2 \xrightarrow{V_\beta} V_3) = V_1 \xrightarrow{V_\beta V_\alpha} V_3$$

  Then G is exact, and the exact structure $\mathcal{E}_G$ coincides with the exact structure $\mathcal{E}_{1,2}$ from example 3.

The notion of reduction of exact structures is closely related to **matrix reductions**. In example 1 above, the exact structure $\mathcal{E}_\alpha$ formed by all short exact sequences that split when restricted to α corresponds to reducing the matrix corresponding to the arrow α. A full sequence of matrix reductions for the quiver Q is given as follows:

$$[A], [B] \longrightarrow \begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}, [B_1\ B_2] \longrightarrow \begin{bmatrix}1 & 0\\ 0 & 0\\ 0 & 0\end{bmatrix}, \begin{bmatrix}0 & 1 & 0\\ B_{12} & 0 & 0\end{bmatrix} \longrightarrow \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}, \begin{bmatrix}0 & 0 & 1 & 0\\ 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}$$

It corresponds to the path of reductions of exact structures from example 3 given by

$$\mathcal{E}_{ab} \to \mathcal{E}_\alpha \to \mathcal{E}_2 \to \mathcal{E}_{min}$$

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Gabriel-Roiter measure on exact categories

We generalize Krause's axiomatic description [Kr] of the Gabriel-Roiter measure on an abelian length category to the context of exact categories. Let therefore $(\mathcal{A}, \mathcal{E})$ be an essentially small exact category, and let $Obj\mathcal{A}$ $(ind\mathcal{A})$ be the set of ismorphism classes of (indecomposable) objects of $\mathcal{A}$.

Definition: A non-zero object S in $(\mathcal{A}, \mathcal{E})$ is $\mathcal{E}-$simple if S admits no $\mathcal{E}-$subobjects except 0 and S, that is, whenever $X \subset_\mathcal{E} S$, then X is the zero object or isomorphic to S.

Definition: An object X of $(\mathcal{A}, \mathcal{E})$ is $\mathcal{E}-$Artinian ($\mathcal{E}-$Noetherian) if any descending (increasing) sequence of $\mathcal{E}-$subobjects of X

$$\rightarrowtail X_n \rightarrowtail X_{n-1} \rightarrowtail \cdots \rightarrowtail X_1 \rightarrowtail X_0$$

becomes stationary. An object X is called $\mathcal{E}-$finite if it is at the same time $\mathcal{E}-$Artinian and $\mathcal{E}-$Noetherian. A finite exact category is an exact category $(\mathcal{A}, \mathcal{E})$ in which any object is $\mathcal{E}-$finite.

Definition: Define the $\mathcal{E}-$length function $l_\mathcal{E} : Obj\mathcal{A} \to \mathbb{N} \cup \{\infty\}$ as supremum over the length of a chain of admissible monics which are not isomorphisms. That is, for an object X of $(\mathcal{A}, \mathcal{E})$, one has $l_\mathcal{E}(X) = n$ if and only if n is the maximal length of a chain of admissible monics which are not isomorphisms

$$0 = X_0 \rightarrowtail X_1 \rightarrowtail \cdots \rightarrowtail X_{n-1} \rightarrowtail X_n = X$$

where $X_i \in Obj\mathcal{A}$ for all i. We say X has finite length if $l_\mathcal{E}(X) < \infty$.

Definition: A *measure for a poset* $\mathcal{S}$ is a morphism of posets $\mu : \mathcal{S} \to \mathcal{P}$ where $(\mathcal{P}, \leq)$ is a totally ordered set.

Theorem

The length function $l_\mathcal{E}$ of an $\mathcal{E}-$finite essentially small exact category $(\mathcal{A}, \mathcal{E})$ is a measure for the poset $Obj\mathcal{A}$.

Lemma (Reduction of length)

If $\mathcal{E}$ and $\mathcal{E}'$ are exact structures on $\mathcal{A}$, such that $\mathcal{E}' \subseteq \mathcal{E}$, then $l_{\mathcal{E}'}(X) \leq l_\mathcal{E}(X)$ for all objects X of $\mathcal{A}$.

Example 2. An $\mathcal{E}-$simple object in an exact category $(\mathcal{A}, \mathcal{E})$ is indecomposable, since the canonical inclusion $X_1 \xrightarrow{i_1} X_1 \oplus X_2$ is admissible in every exact structure $\mathcal{E}$ ([Bü]). Conversely, when $\mathcal{E}$ is the split exact structure $\mathcal{E}_{min}$, then every indecomposable object is $\mathcal{E}_{min}-$simple and $l_{\mathcal{E}_{min}}(X) = 1$. By taking $X \in ind_\mathcal{A}$ where $\mathcal{A} = \text{rep}\, Q$ and $Q = A_3$ as in example 3, we have for $\mathcal{E}_{min} \subseteq \mathcal{E}_{2,3,5} \subseteq \mathcal{E}_{ab}$ that $l_{\mathcal{E}_{min}}(111) = 1 < l_{\mathcal{E}_{2,3,5}}(111) = 2 < l_{\mathcal{E}_{ab}}(111) = 3$.

Let $(\mathcal{A}, \mathcal{E})$ be an $\mathcal{E}-$finite essentially small exact category.
Definition A morphism $\mu_\mathcal{E} : (ind\mathcal{A}, \subset_\mathcal{E}) \to (\mathcal{P}, \leq)$ is called a Gabriel-Roiter measure on the exact category $(\mathcal{A}, \mathcal{E})$ if it verifies the following axioms

(M_1) $\mu_\mathcal{E}$ is a measure

(M_2) $\mu_\mathcal{E}(X) = \mu_\mathcal{E}(Y) \Longrightarrow l_\mathcal{E}(X) = l_\mathcal{E}(Y)\ \forall X, Y \in ind\mathcal{A}$

(M_3) If $l_\mathcal{E}(X) \geq l_\mathcal{E}(Y)$ and $\mu_\mathcal{E}(X') \lneqq \mu_\mathcal{E}(Y)\ \forall X' \subsetneq_\mathcal{E} X$ so $\mu_\mathcal{E}(X) \leq \mu_\mathcal{E}(Y)$.

Theorem: There exist a Gabriel-Roiter measure $\mu_\mathcal{E}$ for $ind\mathcal{A}$.

Proof For a fixed object $X \in ind\mathcal{A}$, we consider the filtrations ending by X

$$F_\mathcal{E}(X) : X_1 \subsetneq_\mathcal{E} \ldots \subsetneq_\mathcal{E} X_n = X$$

where $X_i \in ind\mathcal{A}$ for all i and we consider all vectors

$$l_\mathcal{E}(F_\mathcal{E}(X)) = (l_\mathcal{E}(X_1), \ldots, l_\mathcal{E}(X_n))$$

One can define

$$\mu_\mathcal{E} : ind\mathcal{A} \to (\mathfrak{S}(\mathbb{N}), \lll);$$

$$X \longmapsto \mu_\mathcal{E}(X) = max(l_\mathcal{E}(F_\mathcal{E}(X)))_{F_\mathcal{E}(X)}$$

where $\mathfrak{S}(\mathbb{N})$ is the set of all vectors of natural numbers and $\lll$ is the inverse lexicographic order, so $(\mathfrak{S}(\mathbb{N}), \lll)$ forms a totally ordered set. One can verify that $\mu_\mathcal{E}$ satisfies the three axioms of a Gabriel-Roiter measure.

The poset of exact structures

Definition: Let $\mathcal{A}$ be an additive category. We denote by $(\mathcal{P}_\mathcal{A}, \subseteq)$ the poset of exact structures $\mathcal{E}$ on $\mathcal{A}$, where the partial order is given by inclusion $\mathcal{E}' \subseteq \mathcal{E}$.

Lemma:[Bü] For any additive category $\mathcal{A}$, the sequences isomorphic to

$$A \xrightarrow{\begin{bmatrix}1\\0\end{bmatrix}} A \oplus B \xrightarrow{[0 1]} B$$

form an exact structure $\mathcal{E}_{min}$, called the split exact structure.

Clearly, $\mathcal{E}_{min}$ forms the minimal element in the poset $(\mathcal{P}_\mathcal{A}, \subseteq)$. The following proposition describes the immediate successors of $\mathcal{E}_{min}$ if $\mathcal{A}$ is the category of finite-dimensional modules over a finite-dimensional algebra A:

Proposition

Let A be a finite dimensional algebra and mod$A$ the category of finitely generated modules over A. Let ξ_i be an Auslander-Reiten sequence in mod$A$, then $\mathcal{E}_i = \{X \oplus Y | X \in \mathcal{E}_{min}, Y \in add(\xi_i)\}$ is an exact structure.

Example 3. Consider the category $\mathcal{A} = \text{rep}\, Q$ of representations of the quiver

$$Q : 1 \longrightarrow 2 \longrightarrow 3$$

Then all non-split exact sequences are:

(AR1) $0 \to 010 \to 110 \to 100 \to 0$

(AR2) $0 \to 001 \to 011 \to 010 \to 0$

(AR3) $0 \to 011 \to 111 \oplus 010 \to 110 \to 0$

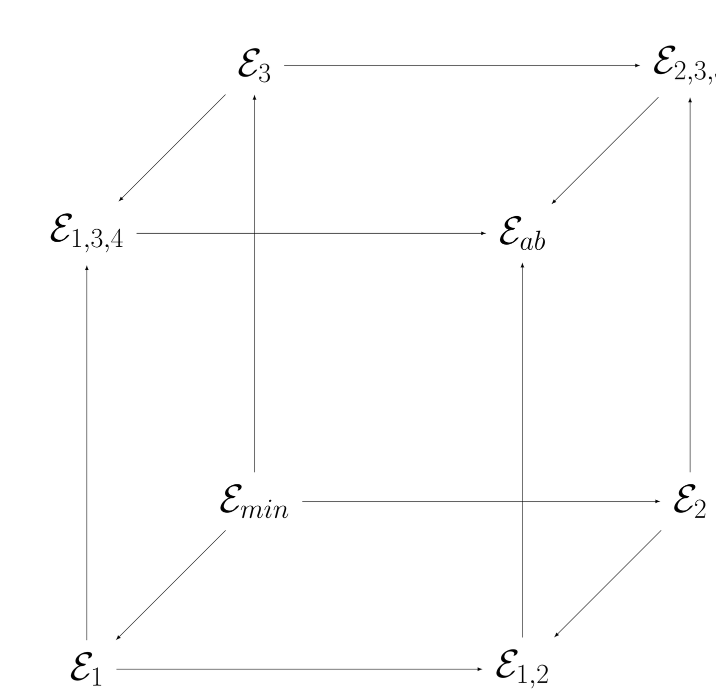
(4) $0 \to 011 \to 111 \to 100 \to 0$

(5) $0 \to 001 \to 111 \to 110 \to 0$

The following list enumerates all exact structures $\mathcal{E}$ on $\mathcal{A}$:

- $\mathcal{E}_{ab}$ is the set of all short exact sequences in $\mathcal{A}$. Thus $(\mathcal{A}, \mathcal{E}_{ab})$ is the abelian structure on the category $\mathcal{A} = \text{rep}\, Q$,
- $\mathcal{E}_{min}$ is the set of all split short exact sequences in $\mathcal{A}$,
- $\mathcal{E}_1 = \{X \oplus Y | X \in \mathcal{E}_{min}, Y \in add(AR1)\}$,
- $\mathcal{E}_2 = \{X \oplus Y | X \in \mathcal{E}_{min}, Y \in add(AR2)\}$,
- $\mathcal{E}_3 = \{X \oplus Y | X \in \mathcal{E}_{min}, Y \in add(AR3)\}$,
- $\mathcal{E}_{1,2} = \mathcal{E}_1 \oplus \mathcal{E}_2 = \{X \oplus Y \mid X \in \mathcal{E}_1, Y \in \mathcal{E}_2\}$,
- $\mathcal{E}_{2,3,5} = \{X \oplus Y \oplus Z | X \in \mathcal{E}_2, Y \in \mathcal{E}_3, Z \in add(5)\}$,
- $\mathcal{E}_{1,3,4} = \{X \oplus Y \oplus Z | X \in \mathcal{E}_1, Y \in \mathcal{E}_3, Z \in add(4)\}$.

Going back to the previous example $\mathcal{E}_{2,3,5} = \mathcal{E}_\alpha$, $\mathcal{E}_{1,3,4} = \mathcal{E}_\beta$ and $\mathcal{E}_{1,2} = \mathcal{E}'$. Hence the poset of exact structures $(\mathcal{P}_\mathcal{A}, \subseteq)$ is described by the following graph, where the oriented arrows present inclusions:



We have for example the following sequence of reductions $\mathcal{E}_{min} \subseteq \mathcal{E}_2 \subseteq \mathcal{E}_{2,3,5} \subseteq \mathcal{E}_{ab}$.

References

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