A McKay correspondence for reflections groups
joint work with Ragnar-Olaf Buchweitz and Colin Ingalls

Eleonore Faber
University of Michigan

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Kleinian singularities

Focus on $n = 2$, and $k = \mathbb{C}$. Then

**Theorem (F. Klein, 1884)**

Let $\Gamma \subseteq SL_2(\mathbb{C})$ be a finite group. Then the quotient singularity $X = \mathbb{C}^2 / \Gamma = \text{Spec}(S^\Gamma)$, i.e., the orbit space of $\Gamma$ acting on $\mathbb{C}^2$, is of the form

$$X = \text{Spec}(\mathbb{C}[x, y, z]/(f)),$$
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$$X = \text{Spec}(\mathbb{C}[x, y, z]/(f)),$$

where $f$ is of type

- $A_n$: $z^2 + y^2 + x^{n+1}$,
- $D_n$: $z^2 + x(y^2 + x^{n-2})$ for $n \geq 4$,
- $E_6$: $z^2 + x^3 + y^4$,
- $E_7$: $z^2 + x(x^2 + y^3)$,
- $E_8$: $z^2 + x^3 + y^5$. 
$A_1$ and $A_2$ – the cone and the cusp

$x^2 + y^2 - z^2 = 0$

$z^2 + y^2 - x^3 = 0$
$A_3$ and $A_4$

$z^2 + y^2 - x^4 = 0$

$z^2 + y^2 - x^5 = 0$
$A_5$ and $A_6$

\[ z^2 + y^2 - x^6 = 0 \]

\[ z^2 + y^2 - x^7 = 0 \]
$D_4 : z^2 + x(y^2 - x^2) = 0$
$D_5$ and $D_6$

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Classical McKay correspondence

$D_7$ and $D_8$

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$E_6 : \quad z^2 + x^3 + y^4 = 0$
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$E_8 : \quad z^2 + x^3 + y^5 = 0$
Let $X$ be a normal surface singularity and let $\pi : \tilde{X} \to X$ be its minimal resolution, with exceptional curves $\bigcup_i E_i$.
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- vertices: $i \leftrightarrow E_i$
- edges: $i - j \leftrightarrow E_i \cap E_j \neq \emptyset$. 

Theorem (Du Val)

The dual resolution resolution graphs of the Kleinian singularities are Coxeter–Dynkin diagrams of type ADE.
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The dual resolution resolution graphs of the Kleinian singularities are Coxeter–Dynkin diagrams of type ADE.
Example: $x^2 + y^2 = z^2$

Dual resolution graph of type $A_1$: 

\[ \pi \]
Example: $z^2 + x(y^2 - x^2) = 0$
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Dual resolution graph of type $D_4$:
McKay correspondence

Let $\Gamma \subseteq SL_2(\mathbb{C})$ be a finite group with irreducible representations $\rho_0, \ldots, \rho_m$:

$\rho_0 = \text{trivial representation}$,

$\rho_1 = c = \text{canonical representation} \quad \Gamma \hookrightarrow GL_2(\mathbb{C})$. 

Observation (J. McKay, 1979): These graphs are extended Coxeter-Dynkin diagrams of type ADE (with arrows in both directions).
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- Arrows: $i \xrightarrow{m_{ij}} j$ iff $\rho_j$ appears with multiplicity $m_{ij}$ in the tensor product representation $c \otimes \rho_i$.
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Definition: $D_4$

The group $\Gamma$ is generated by

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$ 

Five irreps $\rho_i$, four one-dimensional and one two-dimensional $\rho_1 = c$. 

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Example: \( D_4 \)

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\]

Five irreps \( \rho_i \), four one-dimensional and one two-dimensional \( \rho_1 = c \).

The McKay graph:
Thus for $n = 2$ and $\Gamma \in SL_2(\mathbb{C})$:
Have 1-1 correspondence between
- exceptional curves $E_i$ on the minimal resolution of $\mathbb{C}^2/\Gamma$.
- irreducible representations of $\Gamma$ (mod the trivial representation).
- indecomposable projective $\Gamma \ast S = \text{End}_R S$-modules (modulo the trivial module).
- indecomposable $\text{CM}$-modules over $R$ (modulo $R$ itself). [This follows from \textit{Herzog's theorem}, which says that $\text{add}_R(S) = \text{CM}(R)$.]
Theorem (Buchweitz–F–Ingalls)

If $G \subseteq \text{GL}_2(\mathbb{C})$ is a reflection group, let $z = \prod_{s \in \text{reflections}(G)} l_s$ be the hyperplane arrangement and set $\Delta = z^2$. Let further $A = G \ast S$, $e = \frac{1}{|G|} \sum_{g \in G} g$, $\bar{A} = A / AeA$ and $T = S^G$. Then

$$\bar{A} \cong \text{End}_{T/\Delta}(S/z)$$

is a NCR of $T/\Delta$, that is, gldim $\bar{A} = 2$ and $S/z$ is in $\text{CM}(T/\Delta)$. 

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In particular:

$$\text{add}_{T/\Delta}(S/z) = \text{CM}(T/\Delta),$$

i.e., $S/z$ is a $\text{CM}$-representation generator.
The swallowtail: $\Delta$ of $S_4$

$$16x^4z - 4x^3y^2 - 128x^2z^2 + 144xy^2z - 27y^4 + 256z^3 = 0$$
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Here $S/z \simeq T/\Delta \oplus \widetilde{T/\Delta} \oplus \text{syz}(\widetilde{T/\Delta}) \oplus M_{2,0}^2$. 
Questions

- What are the $R$-direct summands of $S/z$?
- Can one describe the $R$-direct summands of $S/z$ for some specific groups, e.g., $S_n$?
- What about the geometry?