On the non-leaving-face property for associahedra

Thomas Brüstle
D. Sleator, R. Tarjan, W. Thurston, 1988:
The associahedron has the non-leaving-face property (NLF): every geodesic connecting two vertices in the graph stays in the minimal face containing both.

A₃
The associahedron (type $A_n$)

- Stasheff polytope
- Tamari lattice
- exchange graph of type $A_n$
The **associahedron** (type $A_n$)

**Definition 1**: vertices = triangulations of $(n+3)$-gon
edges = flips of diagonals

$A_3$ : \[ \begin{array}{c}
\circ \quad \circ \\
\end{array} \]

hexagon
The **associahedron** (type $A_n$)

**Definition 2:** vertices = binary trees with $n+2$ leaves
edges = rotations

$n = 3$:

- $(a ((b c) d)) e$
- $(a (b c)) d e$

Rotation: 

![Diagram showing the associahedron for $n=3$.]
The associahedron (type $A_n$)

**Definition 2:** vertices = binary trees with $n+2$ leaves
edges = rotations

$n=3$:

(a $((bc)d))e$  (a $(bc))d)e$

balanced $O(\log n)$

unbalanced $O(n)$
The associahedron (type $A_n$)

**Definition 3:**
- Vertices = cluster tilting objects type $A_n$
- Edges = mutations

faces given by common summands
The associahedron type $A_n$

**Definition**: vertices = tilting modules over $\Lambda = K(1 \to 2 \to \cdots \to n+1)$

edges = mutations (tilts)
The associahedron type $A_n = \kappa(1 \to 2 \to \cdots \to n)$

**Definition $n+1$:**
- Vertices = torsion classes in mod $A_n$
- Edges = edges of the Hasse quiver given by inclusion
**Question:** What is \( \text{diam}(A_n) = \max \text{ dist}(T, T') \)?

\( T, T' \) triangulation
tree
tilting object
torsion class

**Observe:**

\[
\begin{array}{cccccccccccccccc}
& n & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 \\
\text{diam}(A_n) & 1 & 2 & 4 & 5 & 7 & 9 & 11 & 12 & 15 & 16 & 18 & 20 & 22 \\
\end{array}
\]

**Note:** \( \text{diam}(A_n) > \text{ dist}(B, B[1]) \) for any \( B = \text{End } T \)

max. green
sequences
**Question:** What is $\text{diam}(A_n) = \max \text{ dist}(T, T')$?

**Observe:**

<table>
<thead>
<tr>
<th>$n$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
<th>13</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\text{diam}(A_n)$</td>
<td>1</td>
<td>2</td>
<td>4</td>
<td>5</td>
<td>7</td>
<td>9</td>
<td>11</td>
<td>12</td>
<td>15</td>
<td>16</td>
<td>18</td>
<td>20</td>
<td>22</td>
</tr>
</tbody>
</table>

**STT, 1988:** $\text{diam}(A_n) = 2n - 4$ for $n \gg 0$

**Pournin, 2014:** $\text{diam}(A_n) = 2n - 4$ for $n \geq 10$
**Question:** What is \( \text{diam}(A_n) = \max \text{dist}(T, T')? \)

**Observe:**

<table>
<thead>
<tr>
<th>( n )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
<th>13</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \text{diam}(A_n) )</td>
<td>1</td>
<td>2</td>
<td>4</td>
<td>5</td>
<td>7</td>
<td>9</td>
<td>11</td>
<td>12</td>
<td>15</td>
<td>16</td>
<td>18</td>
<td>20</td>
<td>22</td>
</tr>
</tbody>
</table>

**SST, 1988:** \( \text{diam}(A_n) = 2n - 4 \) for \( n \gg 0 \)

**Pournin, 2014:** \( \text{diam}(A_n) = 2n - 4 \) for \( n \geq 10 \)

**Easy:** \( \text{diam}(A_n) \leq 2n \)
C. Caballos, V. Piland, 2014:

NLT for types $B,C$ (=cyclobedron), $D, E_6, F_4, H_3, H_4$.

Moreover, the diameter in type $D_n$ is $2n-2$. 
C. Caballois, V. Pilaud, 2014:

NLT for types B, C (= cyclohexan), D, E, F₄, H₂, H₄.
Moreover, the diameter in type Dₙ is 2n−2

Y. Lebrun, J.-F. Marceau, 2014:
NLF property is key ingredient to study diameter:

\( T, T' \) share \( U \) \( \iff \) \( T, T' \in F_u \) face of \( U \)

\( \text{NLF: } \ dist_{A_n}(T, T') = dist_{T_u}(T, T') \) if \( T, T' \in F_u \)
**STT:** Key ingredient to prove NLF is a projection $T \mapsto p_u(T)$ onto the face $F_u$ satisfying

(P1) $p_u(T) = T$ for $T \in F_u$

(P2) $F_u \ni T \frac{\text{edge}}{\text{edge}} T' \implies p_u(T') = T$

(P3) $\text{dist}(T, T') \leq 1 \implies \text{dist}(p_u(T), p_u(T')) \leq 1$

This proves NLF:

Given $U \ni T \overset{T'}{\longrightarrow} T'' \cdots \overset{T'}{\longrightarrow} T \in U$, apply $p_u$

$T \overset{T'}{\longrightarrow} T \in F_u$
**STT-projection:**

- Choose an orientation on $U$
- For any $v \in T$, if it does not intersect $U$, keep it.
- If $v$ intersects $U$, drag it along with $U$
- Add $U$ if needed
Caldero, Chapoton, Schiffler, 2004:
Interpret diagonals of \((n+3)\)-gon as indecomposable objects of \(\mathcal{C} = \text{cluster category of type } A_n\).
Caldero, Chapoton, Schiffke, 2004:
Interpret diagonals of \((n+3)\)-gon as indecomposable objects of \(\mathcal{C} = \text{cluster category of type } A_n\).
Caldero, Chapoton, Schiffler, 2004:
Interpret diagonals of \((n+3)\)-gon as indecomposable objects of \(\mathcal{C} = \text{cluster category of type } A_n\).

\[
\begin{align*}
\text{right } F_u \text{-approximation of } v & \neq P_u(v) \\
\end{align*}
\]
The good news: (B, Marceau 2014)
The right $F_n$-approximation satisfies (P1), (P2), (P3)
(we verified this in types $A_n$ and $D_n$, confirming $NLF$ in these cases)
Generalizations:

(1) N. Williams, 2015:

NLF holds for generalized associahedra & permutahedra
defined by a finite Coxeter system
Generalizations:

1. N. Williams, 2015:

   NLF holds for generalized associahedra, permutahedra defined by a finite Coxeter system.

2. B. Marceau, Zhang, 2015:

   NLF holds for triangulations of a marked surface without punctures.
Generalizations:

(1) N. Williams, 2015:

NLF holds for generalized associahedra, permutahedra defined by a finite Coxeter system.

(2) B. Marceau, Zhang, 2015:

NLF holds for triangulations of a marked surface without punctures.

(3) B. Treffinger, Wong, in progress:

A rigid object in a cluster category. Then \( F_u = \frac{1}{u(1)} \), \( RA_u(T) = B_u(T) \) min. right approx. Bourgade completion (jaco) and (P1), (P2) holds.