Standard Auslander-Reiten components of a Krull-Schmidt category

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A : finite dimensional $k$-algebra with $\bar{k} = k$. 

The classical setting

- $A$: finite dimensional $k$-algebra with $\bar{k} = k$. 

- $\text{mod } A$: category of fin dim left $A$-modules. 

- Want to describe maps in $\text{mod } A$ between indecomposables. 

- One introduces Auslander-Reiten quiver $\Gamma_{\text{mod } A}$. 

- In general, $\Gamma_{\text{mod } A}$ describes maps not in $\text{rad } \infty (\text{mod } A)$. 

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Standard components in a module category

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3) (Ringel) \( \Gamma \) is preprojective or preinjective.
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1) If $\Gamma$ is standard, then all but finitely many $\tau$-orbits in $\Gamma$ are periodic.

2) If $\Gamma$ is regular and standard, then $\Gamma$ is stable tube or $\Gamma \cong \mathbb{Z}\Delta$, where $\Delta$ a finite acyclic quiver.
Let $\mathcal{A}$ an additive category with $f : X \to Y$. 

Definition 1: $f$ is a source morphism provided $f$ is not a section, any non-section $g : X \to M$ factors through $f$, if $h : Y \to Y$ with $f = hf$, then $h$ is an automorphism.

Definition 2: In the dual situation, $f$ is a sink morphism.
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2. In dual situation, $f$ is *sink morphism*. 
Definition

A sequence of morphisms $X \xrightarrow{f} Y \xrightarrow{g} Z$ in $\mathcal{A}$ is called \textit{almost split sequence} provided

1. $Y \neq 0$,
2. $f$ is a source morphism, and pseudo-kernel of $g$,
3. $g$ is a sink morphism, and pseudo-cokernel of $f$.

Remark. The above notion unifies almost split sequences in abelian categories and almost split triangles in triangulated categories.
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- **vertices**: the non-isomorphic indecomposables in $\mathcal{A}$.
- **arrows**: given $X, Y$, the number of arrows $X \to Y$ is $d_{X,Y}$.
- **translation**: if $X \to Y \to Z$ almost split, then $\tau Z = X$. 
Objective

Question

1. How to decide a component of $\Gamma_A$ is standard?

2. Are there new types of standard components?

3. We consider these problems for components with a section.
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Let \((\Gamma, \tau)\) be connected translation quiver.
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**Definition**

A connected full subquiver \(\Delta\) of \(\Gamma\) is *section* if

1. \(\Delta\) contains no oriented cycle,
2. \(\Delta\) meets each \(\tau\)-orbit in \(\Gamma\) exactly once,
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Example

Consider a \textit{finite wing} as follows:
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The two longest paths are sections.
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Construct translation quiver $\mathbb{Z}\Delta$ in canonical way.
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**Remark**

For $i \in \mathbb{Z}$, the subquiver $(\Delta, i)$ is section of $\mathbb{Z}\Delta$. 
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For $i \in \mathbb{Z}$, the subquiver $(\Delta, i)$ is section of $\mathbb{Z}\Delta$.

**Notation**

- $\mathbb{N}\Delta = \langle (x, i) \mid x \in \Delta_0, i \in \mathbb{N} \rangle \subseteq \mathbb{Z}\Delta$. 
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- $\mathbb{N}^-\Delta = \langle (x, -i) \mid x \in \Delta_0, i \in \mathbb{N} \rangle \subseteq \mathbb{Z}\Delta$. 
The translation quiver $\mathbb{Z}A_\infty$ is as follows:

\[
\begin{array}{c}
\cdots \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \cdots \\
\downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \\
\circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \\
\downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \\
\cdots \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \cdots
\end{array}
\]
If \( A_\infty^+ \) denotes a right infinite path

\[
\circ \rightarrow \circ \rightarrow \cdots \rightarrow \circ \rightarrow \cdots,
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$$
\circ \rightarrow \circ \rightarrow \cdots \rightarrow \circ \rightarrow \cdots
$$

then $N A_{\infty}^+$ is follows:

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\circ \longleftarrow \cdots \longleftarrow \circ \longleftarrow \cdots
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\[ \circ \longleftarrow \cdots \longleftarrow \circ \longleftarrow \cdots \]

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Example

If \( \mathbb{A}_{\infty} \) denotes a left infinite path

\[ \cdots \rightarrow \circ \rightarrow \circ \rightarrow \cdots \rightarrow \circ \rightarrow \circ, \]
Example

If $\mathbb{A}_\infty^-$ denotes a left infinite path

\[ \cdots \rightarrow \bigcirc \rightarrow \bigcirc \rightarrow \cdots \rightarrow \bigcirc \rightarrow \bigcirc, \]

then $\mathbb{N}^- \mathbb{A}_\infty^-$ is as follows:

\[ \cdots \bigcirc \leftarrow \cdots \bigcirc \leftarrow \bigcirc \leftarrow \cdots \bigcirc \leftarrow \bigcirc \leftarrow \cdots \bigcirc \leftarrow \bigcirc \leftarrow \cdots \bigcirc \leftarrow \bigcirc \leftarrow \cdots \bigcirc \leftarrow \bigcirc \leftarrow \cdots \bigcirc \leftarrow \bigcirc \leftarrow \cdots \bigcirc \leftarrow \bigcirc \leftarrow \cdots \bigcirc \leftarrow \bigcirc \leftarrow \cdots \bigcirc \leftarrow \bigcirc \leftarrow \cdots \bigcirc \leftarrow \bigcirc \leftarrow \cdots \]
Let $\Gamma$ be component of $\Gamma_A$ with a section $\Delta$. 

Properties of components with sections

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**Proposition**

1. Each object in $\Gamma$ uniquely written as $\tau^n X$ with $n \in \mathbb{Z}$, $X \in \Delta$. 

\[ \Delta^+ = \langle \tau^{-n} X | n \rangle_{\mathbb{N}}, X \in \Delta \] 

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1. *Each object in $\Gamma$ uniquely written as $\tau^n X$ with $n \in \mathbb{Z}$, $X \in \Delta$.*

2. *$\Gamma$ embeds $\mathbb{Z}\Delta$, by means of $\tau^n x \mapsto (-n, x)$.***
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2. $\Delta^- = \langle \tau^nX \mid n > 0, X \in \Delta \rangle \subseteq \Gamma$. 
Let $\Gamma$ be component of $\Gamma_A$. 

$\Gamma$ is stable if $\tau X, \tau - X \in \Gamma$, for any $X \in \Gamma$.

$\Gamma$ is $\tau$-periodic if every $X \in \Gamma$ is $\tau$-periodic.

Theorem: If $\Gamma$ is stable, then $\Gamma$ is $\tau$-periodic or $\Gamma \simeq \mathbb{Z} \Delta$ with $\Delta$ acyclic quiver.
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1. $\text{add}(\Delta) \sim = k \Delta$
2. $\text{Hom}_A(\Delta^+, \Delta \cup \Delta^-) = 0$
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General criterion for standardness

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If $\Gamma$ is wing or $\mathbb{Z}A_\infty$, $NA_\infty^+$, $N^-A_\infty^-$, then
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If $\Gamma$ is wing or $\mathbb{ZA}_\infty$, $\mathbb{NA}^+_\infty$, $\mathbb{N^-A^-}_\infty$, then

$\Gamma$ is standard $\iff$ the quasi-simple objects are orthogonal bricks.
Setting

$Q$: connected quiver, which is locally finite, and interval-finite (for all $x, y \in Q_0$, the number of $x \to y$ is finite).

$P_x$: indecomposable projective representation of $Q$ at $x$.

$I_x$: indecomposable injective representation of $Q$ at $x$.

$\text{proj}(Q)$: additive category of the $P_x$, $x \in Q_0$. 

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A representation $M$ of $Q$ is \textit{finitely presented} if

\[ 0 \to P_1 \to P_0 \to M \to 0, \]

where $P_0, P_1 \in \text{proj}(Q)$. 
A representation $M$ of $Q$ is \emph{finitely presented} if there exists a projective resolution

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rep^{+}(Q): finitely presented representations of $Q$. 
rep^+(Q): finitely presented representations of Q.

**Proposition**

rep^+(Q) is Hom-finite, hereditary, abelian.
A component $\Gamma$ of $\Gamma_{\text{rep}^+(Q)}$ is called

1. **preprojective** if $\Gamma$ contains some of the $P_x$.

2. **preinjective** if $\Gamma$ contains some of the $I_x$.

3. **regular** if $\Gamma$ contains none of the $P_x$, $I_x$. 
Classes of AR-components

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Classes of AR-components

**Definition**

A component \( \Gamma \) of \( \Gamma_{\text{rep}^+}(Q) \) is called

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Theorem

Let $Q$ connected, strongly locally finite.
Theorem

Let $Q$ connected, strongly locally finite.

1. The unique preprojective component of $\Gamma_{\text{rep}^+(Q)}$ is standard and embeds in $\mathbb{N}Q^{-1}$.

Proof.

The $P_{\mathbf{x}}, \mathbf{x} \in Q_0$, form subquiver $\Delta \cong Q^{-1}$.

$\exists!$ preprojective component $\Delta$ of which $\Delta$ is section.

$\Delta^- = \emptyset$ and $\Delta^+$ no left-$\infty$ path.

Add $(\Delta)$ $\cong kQ^{-1}$ and $\text{Hom}(\Delta^+, \Delta) = 0$. 

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Theorem

Let $Q$ connected, strongly locally finite.

1. The unique preprojective component of $\Gamma_{\text{rep}^+(Q)}$ is standard and embeds in $\mathbb{N}Q^\text{op}$.

2. The preinjective components of $\Gamma_{\text{rep}^+(Q)}$ are all standard, and embed in $\mathbb{N}^{-}Q^\text{op}$.
Preprojective component and preinjective components

Theorem

Let $Q$ connected, strongly locally finite.

1. The unique preprojective component of $\Gamma_{\text{rep}^+(Q)}$ is standard and embeds in $\mathbb{N}Q^\text{op}$.

2. The preinjective components of $\Gamma_{\text{rep}^+(Q)}$ are all standard, and embed in $\mathbb{N}^\text{−}Q^\text{op}$.

Proof. The $P_x, x \in Q_0$, form subquiver $\Delta \cong Q^\text{op}$.
Theorem

Let $Q$ connected, strongly locally finite.

1. The unique preprojective component of $\Gamma_{\text{rep}^+(Q)}$ is standard and embeds in $\mathbb{N}Q^\text{op}$.

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$\exists!$ preprojective component $\mathcal{P}$ of which $\Delta$ is section.
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Let $Q$ connected, strongly locally finite.

1. The unique preprojective component of $\Gamma_{\text{rep}^+(Q)}$ is standard and embeds in $\mathbb{N}Q^{\text{op}}$.

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$\Delta^{-} = \emptyset$ and $\Delta^{+}$ no left-$\infty$ path.

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Theorem

Let $Q$ connected, infinite, strongly locally finite.
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1. The regular components of $\Gamma_{\text{rep}^+(Q)}$ are wings or $\mathbb{Z}A_\infty, NA_\infty^+, NA_\infty^-$.
Theorem

Let $Q$ connected, infinite, strongly locally finite.

1. The regular components of $\Gamma_{\text{rep}^+}(Q)$ are wings or $\mathbb{Z}A_\infty, NA_\infty^+, NA_\infty^-.$

2. The regular components are all standard $\iff Q$ of infinite Dynkin types $A_\infty, A_\infty^+, D_\infty.$
Theorem

Let $Q$ be infinite Dynkin quiver.
Infinite Dynkin case

**Theorem**

Let $Q$ be infinite Dynkin quiver.

1. $\Gamma_{\text{rep}^+}(Q)$ has at most four components, at most two regular, all standard.
Let $Q$ be infinite Dynkin quiver.

1. $\Gamma_{\text{rep}^+(Q)}$ has at most four components, at most two regular, all standard.

2. Wings, $\mathbb{Z}A_\infty$, $NA_\infty^+$, $N^-A_\infty^-$ all appear in this setting.
Let $Q$ be connected, strongly locally finite.
The derived category $D^b(\text{rep}^+(Q))$

1. Let $Q$ be connected, strongly locally finite.
2. $D^b(\text{rep}^+(Q))$ is Hom-finite, Krull-Schmidt.
Let $Q$ be connected, strongly locally finite.

2. $\mathcal{D}^b(\text{rep}^+(Q))$ is Hom-finite, Krull-Schmidt.

3. $\Gamma_{\mathcal{D}^b(\text{rep}^+(Q))}$ has a connecting component $C_Q$, containing
The derived category $D^b(\text{rep}^+(Q))$

1. Let $Q$ be connected, strongly locally finite.
2. $D^b(\text{rep}^+(Q))$ is Hom-finite, Krull-Schmidt.
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   - the preprojective component of $\Gamma_{\text{rep}^+(Q)}$. 

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1. Let $Q$ be connected, strongly locally finite.

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3. $\Gamma_{D^b(\text{rep}^+(Q))}$ has a connecting component $\mathcal{C}_Q$, containing
   - the preprojective component of $\Gamma_{\text{rep}^+(Q)}$.
   - shift by -1 of all preinjective components of $\Gamma_{\text{rep}^+(Q)}$. 
Theorem

Let $Q$ be connected, strongly locally finite.
Standard components in $D^b(\text{rep}^+(Q))$

**Theorem**

Let $Q$ be connected, strongly locally finite.

1. $C_Q$ is standard and embeds in $\mathbb{Z}Q^{\text{op}}$.
Theorem

Let $Q$ be connected, strongly locally finite.

1. $C_Q$ is standard and embeds in $\mathbb{Z}Q^{\text{op}}$.
2. $Q$ no infinite path $\Rightarrow C_Q \cong \mathbb{Z}Q^{\text{op}}$. 
Standard components in $D^b(\text{rep}^+(Q))$

**Theorem**

Let $Q$ be connected, strongly locally finite.

1. $\mathcal{C}_Q$ is standard and embeds in $\mathbb{Z}Q^{\text{op}}$.
2. $Q$ no infinite path $\Rightarrow \mathcal{C}_Q \cong \mathbb{Z}Q^{\text{op}}$.
3. $Q$ of infinite Dynkin type $\Rightarrow \Gamma_{D^b(\text{rep}^+(Q))}$ has at most 3 components up to shift, all standard.
Let $A$ be finite dimensional $k$-algebra.
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Let $\Gamma$ be component of $\Gamma_{\text{mod}A}$.
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Let $\Gamma$ be component of $\Gamma_{\text{mod}A}$.

**Theorem**

1. If $\Gamma$ has a section $\Delta$, then it is standard
   \[ \iff \text{Hom}_A(X, \tau Y) = 0 \quad \text{for} \quad X, Y \in \Delta. \]
Let $A$ be finite dimensional $k$-algebra. Let $\Gamma$ be component of $\Gamma_{\text{mod}A}$.

**Theorem**

1. If $\Gamma$ has a section $\Delta$, then it is standard $\iff \text{Hom}_A(X, \tau Y) = 0$ for $X, Y \in \Delta$.
2. $\Gamma$ is standard with a section $\iff \Gamma$ is a connecting component of AR-quiver of a tilted factor algebra of $A$. 