Invariant Theory of AS-Regular Algebras: A Survey

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Invariants under $S_n$
Permutations of $x_1, \cdots, x_n$. 

(Painter: Christian Albrecht Jensen) (Wikipedia)
Gauss’ Theorem

The subring of invariants under $S_n$ is a polynomial ring

$$k[x_1, \cdots, x_n]^{S_n} = k[\sigma_1, \cdots, \sigma_n]$$

where $\sigma_\ell$ are the $n$ elementary symmetric functions for $\ell = 1, \ldots, n$, or the $n$ power sums:

$$P_\ell = x_1^\ell + \cdots + x_i^\ell + \cdots + x_n^\ell$$

Question: When is $k[x_1, \cdots, x_n]^G$ a polynomial ring? ($G$ a finite group of graded automorphisms.)
Shephard-Todd-Chevalley Theorem

Let $k$ be a field of characteristic zero.

**Theorem (1954).** The ring of invariants $k[x_1, \cdots, x_n]^G$ under a finite group $G$ is a polynomial ring if and only if $G$ is generated by reflections.

A linear map $g$ on $V$ is called a **reflection** of $V$ if all but one of the eigenvalues of $g$ are 1, i.e. $\dim V^g = \dim V - 1$.

**Example:** Transposition permutation matrices are reflections, and $S_n$ is generated by reflections.
When is $k[x_1, x_2, \ldots, x_n]^G$:

- A Gorenstein ring? Watanabe’s Theorem (1974), Stanley’s Theorem (1978) ($H_{A^G}(t^{-1}) = \pm t^m H_{A^G}(t)$).

**Example.** Let $g = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ act on $k[x, y]$

$$k[x, y]^g = k\langle x^2, xy, y^2 \rangle \cong \frac{k[a, b, c]}{\langle b^2 - ac \rangle}, \quad H(t) = \frac{1 + t^2}{(1 - t^2)^2}$$

Noncommutative Generalizations

Replace $k[x_1, \cdots, x_n]$ by a connected graded noetherian Artin-Schelter regular algebra $A$. Let $k = \mathbb{C}$.

$G$ a group of graded automorphisms of $A$. Not all linear maps act on $A$.

**Question:** Under what conditions on $G$ is $A^G$ Artin-Schelter regular, or AS-Gorenstein, or a “complete intersection”?

More generally, consider finite dimensional (semisimple) Hopf algebras $H$ acting on $A$. 
Noetherian connected graded algebra $A$ is Artin-Schelter Gorenstein if:

- $A$ has graded injective dimension $n < \infty$ on the left and on the right,
- $\Ext^i_A(k, A) = \Ext^i_{A^\op}(k, A) = 0$ for all $i \neq n$, and
- $\Ext^n_A(k, A) \cong \Ext^n_{A^\op}(k, A) \cong k(\ell)$ for some $\ell$.

If in addition,

- $A$ has finite (graded) global dimension, and
- $A$ has finite Gelfand-Kirillov dimension,

then $A$ is called Artin-Schelter regular of dimension $n$.

An Artin-Schelter regular graded domain $A$ is called a quantum polynomial ring of dimension $n$ if $H_A(t) = (1 - t)^{-n}$. 

\[ H_A(t) = (1 - t)^{-n} \]
Graded automorphisms of $\mathbb{C}_q[x, y]$

If $q \neq \pm 1$ there are only diagonal automorphisms:

$$g = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}.$$

When $q = \pm 1$ there also are automorphisms of the form:

$$g = \begin{bmatrix} 0 & a \\ b & 0 \end{bmatrix}:

yx = qxy

\begin{align*}
g(yx) &= g(qxy) \\
axby &= qbyax \\
abxy &= q^2 abxy
\end{align*}

q^2 = 1.
Noncommutative Gauss’ Theorem?

**Example:** \( S_2 = \langle g \rangle \), for \( g = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \), acts on \( A = \mathbb{C}_{-1}[x, y] \) and \( A^{S_2} \) is generated by

\[
P_1 = x + y \text{ and } P_2 = x^3 + y^3
\]

\( (x^2 + y^2 = (x + y)^2 \) and \( g \cdot xy = yx = -xy \) so no generators in degree 2). The generators are NOT algebraically independent. \( A^{S_2} \) is NOT AS-regular (but it is a hyperplane in an AS-regular algebra).

The transposition \((1, 2)\) is NOT a “reflection”.
Definition of “reflection”: Want $A^G$ AS-regular

All but one eigenvalue of $g$ is 1 $\sim$

The trace function of $g$ acting on $A$ of dimension $n$ has a pole of order $n - 1$ at $t = 1$, where

$$\text{Tr}_A(g, t) = \sum_{k=0}^{\infty} \text{trace}(g|A_k)t^k = \frac{1}{(1 - t)^{n-1}q(t)}$$ for $q(1) \neq 0.$

Conjecture: $A^G$ is AS-regular if and only if $G$ is generated by “reflections”.
Examples $G = \langle g \rangle$ on $A = \mathbb{C}_{-1}[x, y]$ ($yx = -xy$):

(a) $g = \begin{bmatrix} \epsilon_n & 0 \\ 0 & 1 \end{bmatrix}$, $Tr(g, t) = \frac{1}{(1 - t)(1 - \epsilon_n t)}$, $A^G$ AS-regular.

(b) $g = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $Tr(g, t) = \frac{1}{1 + t^2}$, $A^G$ not AS-regular.

(c) $g = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, $Tr(g, t) = \frac{1}{(1 - t)(1 + t)}$, $A^G$ AS-regular.

$A^G = \mathbb{C}[xy, x^2 + y^2]$.

For $A = \mathbb{C}_{q_{ij}}[x_1, \cdots, x_n]$ the groups generated by “reflections” are exactly the groups whose fixed rings are AS-regular rings.
What are the reflection groups?

For quantum polynomial rings they must be generated by classical reflections and “mystic” reflections.

Example: The “binary dihedral groups” of order $4\ell$ generated by

$$g_1 = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \text{ and } g_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

for $\lambda$ a primitive $2\ell$th root of unity, acts on $A = \mathbb{C}_{-1}[x, y]$.

$$A^G = \mathbb{C}[xy, x^{2\ell} + y^{2\ell}]$$
Molien’s Theorem: Using trace functions

Jørgensen-Zhang: \[ H^G_A(t) = \frac{1}{|G|} \sum_{g \in G} \operatorname{Tr}_A(g, t) \]

Example (c) \( A = \mathbb{C}_1[x, y] \) and \( g = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \)

\( \sigma_1 = x^2 + y^2, \sigma_2 = xy \) and \( A^G \cong \mathbb{C}[^0_1, \sigma_2] \).

\[ H^G_A(t) = \frac{1}{4(1-t)^2} + \frac{2}{4(1-t^2)} + \frac{1}{4(1+t)^2} = \frac{1}{(1-t^2)^2}. \]
Theorem (Chevalley-Serre). If $G$ acts on $A = \mathbb{C}[x_1, \ldots, x_n]$ with \( \theta_i \) a set of $n$ homogeneous algebraically independent $G$-invariants of $\mathbb{C}[x_1, \ldots, x_n]$, and if $I = \langle \theta_1, \ldots, \theta_n \rangle$, then $A/I$, as a $G$-module, is isomorphic to $t$ copies of the regular representation of $G$, where

\[
t = \prod_i \frac{\deg(\theta_i)}{|G|}
\]

(when $G$ is generated by reflections then $t = 1$).
Theorem. Let $A$ be AS-regular of GKdim $A = n$ with Hilbert series $1/((1 - t)^n p(t))$. If there are $n$ homogeneous $G$-invariant elements $\theta_i$ with $\theta_i$ normal in $A$ and $\theta_i$ regular on $A/\langle \theta_1, \ldots, \theta_{i-1} \rangle$, then for $I = \langle \theta_1, \ldots, \theta_n \rangle$ as a $G$-module, $A/I$ is isomorphic to $t$ copies of the regular representation, where

$$t = \prod_i \frac{\deg(\theta_i)}{|G|(p(1))}$$

(when $G$ is generated by reflections then $t = 1$).
Example 1. Binary dihedral groups on \( A = \mathbb{C}_-1[x, y] \) with

\[
A^G = \mathbb{C}[xy, x^{2\ell} + y^{2\ell}].
\]

\( \mathbb{C}_-1[x, y]/(xy, x^{2\ell} + y^{2\ell}) \) is one copy of regular representation of \( G \).

Example 2. \( S_n \) acting on \( A = \mathbb{C}_-1[x_1, \ldots, x_n] \) with \( \theta_i \) the ith symmetric function in the \( \{x_i^2\} \) – e.g. \( n=2 \)

\( \mathbb{C}_-1[x, y]/\langle x^2 + y^2, x^2 y^2 \rangle \) is \( (2 \cdot 4)/2 = 4 \) copies of the regular representation of \( S_2 \).
Let \((H, \Delta, \epsilon, S)\) be a Hopf algebra and \(A\) be a Hopf-module algebra so
\[
h \cdot (ab) = \sum (h_1 \cdot a)(h_2 \cdot b) \quad \text{and} \quad h \cdot 1_A = \epsilon(h)1_A
\]
for all \(h \in H\), and all \(a, b \in A\).
The invariants of \(H\) on \(A\) are
\[
A^H := \{a \in A \mid h \cdot a = \epsilon(h)a \text{ for all } h \in H\}.
\]
When \(H = k[G]\) and \(\Delta(g) = g \otimes g\) then \(g \cdot (ab) = g(a)g(b)\).
Etingof and Walton (2013): Let $H$ be a finite dimensional semisimple Hopf algebra over a field of characteristic zero, and let $A$ be a commutative domain. If $A$ is an $H$-module algebra for an inner faithful action of $H$ on $A$, then $H$ is a group algebra.

**Question:** Under what conditions on $H$ is $A^H$ an AS-regular algebra?

When is $H$ a “quantum reflection group”?
Kac/Masuoka’s 8-dimensional semisimple Hopf algebra

$H_8$ is generated by $x, y, z$ with the following relations:

\[
x^2 = y^2 = 1, \quad xy = yx, \quad zx = yz,
\]

\[
zy = xz, \quad z^2 = \frac{1}{2}(1 + x + y - xy).
\]

\[
\Delta(x) = x \otimes x, \quad \Delta(y) = y \otimes y,
\]

\[
\Delta(z) = \frac{1}{2}(1 \otimes 1 + 1 \otimes x + y \otimes 1 - y \otimes x)(z \otimes z),
\]

\[
\epsilon(x) = \epsilon(y) = \epsilon(z) = 1, \quad S(x) = x^{-1}, \quad S(y) = y^{-1}, \quad S(z) = z.
\]
\( H_8 \) has a unique irreducible 2-dimensional representation on \( \mathbb{C}u + \mathbb{C}v \) given by

\[
\begin{align*}
x & \mapsto \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \\
y & \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \\
z & \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},
\end{align*}
\]

Example 1: Let \( A = \mathbb{C}\langle u, v \rangle/\langle u^2 - v^2 \rangle \).
\( A^H = \mathbb{C}[u^2, (uv)^2 - (vu)^2] \), a commutative polynomial ring.
\( H \) is “quantum reflection group” for \( A \).

Example 2: Let \( A = \mathbb{C}\langle u, v \rangle/\langle vu - iuv \rangle \).
\( A^H = \mathbb{C}[u^2v^2, u^2 + v^2] \), a commutative polynomial ring.
\( H \) is “quantum reflection group” for \( A \).
$H$ not semisimple

The Sweedler algebra $H(-1)$ generated by $g$ and $x$

\[ g^2 = 1, \quad x^2 = 0, \quad xg = -gx \]

\[ \Delta(g) = g \otimes g \quad \Delta(x) = g \otimes x + x \otimes 1, \]

\[ \epsilon(g) = 1, \epsilon(x) = 0 \quad S(g) = g, \quad S(x) = -gx. \]

Then $H(-1)$ acts on $k[u, v]$ as

\[ x \mapsto \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad g \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \]

\[ k[u, v]^{H(-1)} = k[u, v^2]. \]
Questions:

When is $A^H$ regular?

Are the trace functions useful in understanding when $H$ is a “quantum reflection group”? What are the elements whose traces determine if $H$ is a “quantum reflection group”?
Gorenstein Invariant Subrings
Watanabe’s Theorem (1974):

If $G$ is a finite subgroup of $SL_n(k)$ then $k[x_1, \cdots, x_n]^G$ is Gorenstein.

If $A$ is AS-regular, when is $A^G$ AS-Gorenstein?

What is the generalization of determinant $= 1$?
Trace Functions and Homological Determinant

When $A$ is AS-regular of dimension $n$, then when the trace is written as a Laurent series in $t^{-1}$

$$Tr_A(g, t) = (-1)^n(hdet g)^{-1} t^{-\ell} + \text{higher terms}$$

(Jing-Zhang)

**Generalized Watanabe’s Theorem** (Jørgensen-Zhang): $A^G$ is AS-Gorenstein when all elements of $G$ have homological determinant 1.
If \( g \) is a 2-cycle and \( A = \mathbb{C}_{-1}[x_1 \ldots, x_n] \) then

\[
Tr_A(g, t) = \frac{1}{(1 + t^2)(1 - t)^{n-2}}
\]

\[
= (-1)^n \frac{1}{t^n} + \text{higher terms}
\]

so \( \text{hdet } g = 1 \), and for ALL groups \( G \) of \( n \times n \) permutation matrices, \( A^G \) is AS-Gorenstein. Not true for commutative polynomial ring – e.g.

\[
\mathbb{C}[x_1, x_2, x_3, x_4]^{\langle(1,2,3,4)\rangle}
\]

is not Gorenstein, while

\[
\mathbb{C}_{-1}[x_1, x_2, x_3, x_4]^{\langle(1,2,3,4)\rangle}
\]

is AS-Gorenstein.
Binary Polyhedral Groups

Felix Klein (1884)

Classified the finite subgroups of $SL_2(k)$, for $k$ an algebraically closed field of char 0, and calculated invariants $k[u, v]^G$. 
Actions of Binary Polyhedral Groups on $k[u, v]$

$G$ a finite subgroup of $SL_2(k)$

$k[u, v]^G$ is a hypersurface ring

$k[u, v]^G \cong k[x, y, z]/(f(x, y, z))$, a “Kleinian singularity”, of type A,D or E (corresponding to the type of McKay quiver of the irreducible representations of the group $G$).
The Homological Determinant of a Hopf Action

Since $\text{Ext}_A^n(k, A)$ is 1-dimensional, the left $H$-action on $\text{Ext}_A^n(k, A)$ defines an algebra map $\eta' : H \to k$ such that $h \cdot e = \eta'(h)e$ for all $h \in H$.

The homological determinant $\text{hdet}$ is equal to $\eta' \circ S$, where $S$ is the antipode of $H$.

The homological determinant is trivial if $\text{hdet} = \epsilon$. 
Find all $H$, a finite dimensional Hopf algebra acting on $A$, an AS-regular algebra of dimension 2:

$$k_J[u, v] := k\langle u, v \rangle/(vu - uv - u^2)$$

or

$$k_q[u, v] := k\langle u, v \rangle/(vu - quv),$$

with trivial homological determinant, so that $A$ is an $H$ module algebra, the action is inner faithful and preserves the grading.

Use the classification of finite Hopf quotients of the coordinate Hopf algebra $O_q(SL_2(k))$ (Bichon-Natale, Müller, Stefan).
<table>
<thead>
<tr>
<th>AS reg alg $A$ gldim 2</th>
<th>f.d. Hopf algebra(s) $H$ acting on $A$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$k[u, v]$</td>
<td>$k\tilde{\Gamma}$</td>
</tr>
<tr>
<td>$k_{-1}[u, v]$</td>
<td>$kC_n$ for $n \geq 2$; $kD_{2n}$; $D(\tilde{\Gamma})^\circ$ for $\tilde{\Gamma}$ nonabelian</td>
</tr>
<tr>
<td>$k_q[u, v]$, $q$ root of 1, $q^2 \neq 1$ if $U$ non-simple</td>
<td>$kC_n$ for $n \geq 3$; $(T_{q,\alpha,n})^\circ$; $1 \rightarrow (k\tilde{\Gamma})^\circ \rightarrow H^\circ \rightarrow u_q(sI_2)^\circ \rightarrow 1$; $1 \rightarrow (k\Gamma)^\circ \rightarrow H^\circ \rightarrow u_{2,q}(sI_2)^\circ \rightarrow 1$;</td>
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</tr>
<tr>
<td>$k_q[u, v]$, $q$ root of 1, $q^2 = 1$ if $U$ simple, $o(q)$ odd, and $q^4 \neq 1$</td>
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<tr>
<td>$k_q[u, v]$, $q$ not root 1</td>
<td>$kC_n$, $n \geq 2$</td>
</tr>
<tr>
<td>$k_J[u, v]$</td>
<td>$kC_2$</td>
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</table>
Commutative Complete Intersections

Theorem (Gulliksen) (1971):
Let $A$ be a connected graded noetherian commutative algebra. Then the following are equivalent.

1. $A$ is isomorphic to $k[x_1, x_2, \ldots, x_n]/(d_1, \ldots, d_m)$ for a homogeneous regular sequence.

2. The Ext-algebra $\text{Ext}_A^*(k, k)$ is noetherian.

3. The Ext-algebra $\text{Ext}_A^*(k, k)$ has finite GK-dimension.
Let $A$ be a connected graded finitely generated algebra.

1. We say $A$ is a *classical complete intersection* if there is a connected graded noetherian AS regular algebra $R$ and a sequence of regular normal homogeneous elements $\{\Omega_1, \cdots, \Omega_n\}$ of positive degree such that $A$ is isomorphic to $R/(\Omega_1, \cdots, \Omega_n)$.

2. We say $A$ is a *complete intersection of noetherian type* if the Ext-algebra $\text{Ext}_A^*(k, k)$ is noetherian.

3. We say $A$ is a *complete intersection of growth type* if the Ext-algebra $\text{Ext}_A^*(k, k)$ has finite Gelfand-Kirillov dimension.

4. We say $A$ is a *weak complete intersection* if the Ext-algebra $\text{Ext}_A^*(k, k)$ has subexponential growth.
Noncommutative case:

Classical C.I. \( \Downarrow \) C.I. of Growth Type \( \implies \) Weak C.I.

C.I. of Noetherian Type \( \Downarrow \)
Noncommutative case:

\[ \text{Classical C.I.} \quad \leftarrow \quad \text{C.I. of Noetherian Type} \]

\[ \downarrow \quad \uparrow \]

\[ \text{C.I. of Growth Type} \quad \rightarrow \quad \text{Weak C.I.} \]

\[ \downarrow \]
$A^G$ a complete intersection:

Theorem: (Kac and Watanabe – Gordeev) (1982). If $\mathbb{C}[x_1, \ldots, x_n]^G$ is a complete intersection then $G$ is generated by bi-reflections (all but two eigenvalues are 1).

For an AS-regular algebra $A$ a graded automorphism $g$ is a “bi-reflection” of $A$ if

$$Tr_A(g, t) = \sum_{k=0}^{\infty} \text{trace}(g|A_k)t^k$$

$$= \frac{1}{(1 - t)^{n-2}q(t)},$$

$n = \text{GKdim } A$, and $q(1) \neq 0$. 
Example:

\( A^G \) a complete intersection

\[ A = \mathbb{C}_{-1}[x, y, z] \] is regular of dimension 3, and

\[
g = \begin{bmatrix}
0 & -1 & 0 \\
1 & 0 & 0 \\
0 & 0 & -1
\end{bmatrix}
\]

acts on it. The eigenvalues of \( g \) are \(-1, i, -i\) so \( g \) is not a bi-reflection of \( A_1 \). However,

\[
\text{Tr}_A(g, t) = 1/((1 + t)^2(1 - t)) = -1/t^3 + \text{higher degree terms}
\]

and \( g \) is a “bi-reflection” with \( \text{hdet } g = 1 \).

\[
A^g \cong k[X, Y, Z, W]/\langle W^2 - (X^2 + 4Y^2)Z \rangle,
\]

a commutative complete intersection.
Invariants $A^G$

Classical C.I. \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \qu
Gauss’ Theorem

Invariants of \( \mathbb{C}_{-1}[x_1, \ldots, x_n] \) under the full Symmetric Group \( S_n \):

\( \mathbb{C}_{-1}[x_1, \ldots, x_n]^{S_n} \) and \( \mathbb{C}_{-1}[x_1, \ldots, x_n]^{A_n} \) are classical complete intersections.

Permutations in \( S_n \) are “bi-reflections” if and only if they are 2-cycles or 3-cycles.

**Theorem.** Let \( A = k_{-1}[x_1, \cdots, x_n] \) and \( G \) be a finite subgroup of permutations of \( \{x_1, \cdots, x_n\} \). If \( G \) is generated by quasi-bireflections then \( A^G \) is a classical complete intersection.

**Question:** Is the converse true?
Graded Down-up Algebras

\[ A(\alpha, \beta), \beta \neq 0: \]

Theorem. Let \( A \) be a down-up algebra with \( \beta \neq 0 \)
\[ (y^2x = \alpha yxy + \beta xy^2 \text{ and } yx^2 = \alpha yxy + \beta x^2y) \]
and \( G \) be a finite subgroup of graded automorphisms of \( A \). Then the following are equivalent:

- \( A^G \) is a growth type complete intersection.
- \( A^G \) is cyclotomic Gorenstein and \( G \) is generated by quasi-bireflections.
- \( A^G \) is cyclotomic Gorenstein.

Question: Are these \( A^G \) also classical complete intersections?
Veronese Subrings

For a graded algebra $A$ the $r$th Veronese $A^{(r)}$ is the subring generated by all monomials of degree $r$.

If $A$ is AS-Gorenstein of dimension $n$, then $A^{(r)}$ is AS-Gorenstein if and only if $r$ divides $\ell$ where $\text{Ext}_A^n(k, A) = k(\ell)$ (Jørgensen-Zhang).

Let $g = \text{diag}(\lambda, \cdots, \lambda)$ for $\lambda$ a primitive $r$th root of unity; $G = (g)$ acts on $A$ with $A^{(r)} = A^G$.

If the Hilbert series of $A$ is $(1 - t)^{-n}$ then

$$\text{Tr}_A(g^i, t) = \frac{1}{(1 - \lambda^i t)^n}.$$ 

For $n \geq 3$ the group $G = (g)$ contains no “bi-reflections”, so $A^G = A^{(r)}$ should not be a complete intersection.
Theorem:
Let $A$ be noetherian connected graded algebra.

Suppose the Hilbert series of $A$ is $(1 - t)^{-n}$. If $r \geq 3$ or $n \geq 3$, then $H_{A^{(r)}}(t)$ is not cyclotomic.
Consequently, $A^{(r)}$ is not a complete intersection of any type.
Auslander’s Theorem

Let $G$ be a finite subgroup of $GL_n(k)$ that contains no reflections, and let $A = k[x_1, \ldots, x_n]$. Then the skew-group ring $A \# G$ is isomorphic to $\operatorname{End}_{A^G}(A)$ as rings.

Question: Does Auslander’s Theorem generalize to our context?