On the Homology of the Ginzburg Algebra

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Outline:

Quivers with Potential

Definition (Derksen-Weyman-Zelevinsky)

A **quiver with potential** (QP for short) is a pair (Q, W) where Q is a quiver with

- no loops
- no 2-cycles

and W is a **potential** on Q, i.e., an element of

$$HH_0(kQ) = kQ/[kQ, kQ].$$

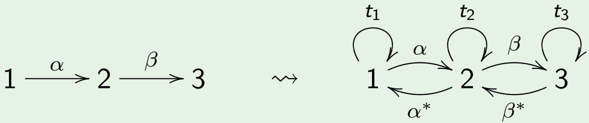
Equivalently, W is a linear combination of cycles of Q considered up to cyclic equivalence.

The Ginzburg Algebra

The **Ginzburg algebra** $\Gamma_{(Q,W)}$ of a QP (Q, W) is the dga constructed as follows. As a graded algebra, $\Gamma_{(Q,W)} = k\widehat{Q}$ where $\widehat{Q}$ is the quiver:

1. Start with Q (in degree 0).
2. Add reversed arrows $\alpha^* : j \to i$ (degree -1) for each $\alpha : i \to j$ in Q.
3. Add loops t_i (degree -2) for each vertex i of Q.

Example



The Ginzburg Algebra

Equipped with a differential d determined by:

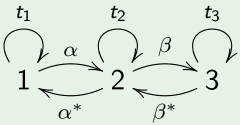
- $d\alpha = 0$ for $\alpha \in Q_1$
- $d\alpha^* = \partial_\alpha W$ for $\alpha \in Q_1$
  where $\partial_\alpha : HH_0(kQ) \to kQ$ (the cyclic partial derivative) given by

$$\partial_\alpha(w) = \sum_{w=u\alpha v} vu$$

- $dt_i = e_i \left(\sum_{\alpha \in Q_1} [\alpha, \alpha^*]\right) e_i$
  $([x, y] = xy - yx)$.
- Extend to all of $\Gamma_{(Q,W)}$ by Leibniz law:

$$d(xy) = d(x)y + (-1)^{|x|} x d(y).$$

Example



$$d\alpha = d\beta = d\alpha^* = d\beta^* = 0$$
$$dt_1 = \alpha\alpha^*, dt_3 = -\beta^*\beta$$
$$dt_2 = \beta\beta^* - \alpha^*\alpha$$

The Ginzburg Algebra

If Q is acyclic (e.g. Q Dynkin) $HH_0(kQ) = 0$; hence the only potential Q admits is the trivial one $W = 0$. In this situation we write $\Gamma_Q = \Gamma_{(Q,0)}$.

For Q acyclic $kQ = H^0\Gamma_Q$. But what about higher degrees?

Definition

Define the **weight** of a path γ in $\widehat{Q}$ to be the number of times γ traverses a loop t_i.

Gives a weight grading on Γ_Q. Descends to a grading on $H^*\Gamma_Q$. Denote the weight w component of $H^*\Gamma_Q$ by $H^*_w\Gamma_Q$.

The Preprojective Algebra

Recall the **preprojective algebra** of Q is the algebra $\Pi_Q = k\overline{Q}/(\rho)$ where

- $\overline{Q}$ is the subquiver of $\widehat{Q}$ consisting of arrows of weight 0.
- $\rho = \sum_{\alpha \in Q_1} [\alpha, \alpha^*]$.

Example

$$1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3 \quad \rightsquigarrow \quad 1 \underset{\alpha^*}{\overset{\alpha}{\rightleftarrows}} 2 \underset{\beta^*}{\overset{\beta}{\rightleftarrows}} 3$$

Π_Q contains kQ as a subalgebra. As a (right) kQ-module it splits into a direct sum of preprojective indecomposable modules with each isoclass represented exactly once.

$\Pi_Q = H_0^*\Gamma_Q \subset H^*\Gamma_Q$. This inclusion is proper in general.

The Preprojective Algebra

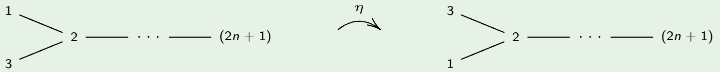
For Q Dynkin, we have an (covariant) involution η of Π_Q:

1. Let $\bar{\eta}$ be the involution of the underlying graph $|Q|$ of Q

$$\bar{\eta} = \begin{cases} \text{identity map} & |Q| = D_{2n}, E_7, E_8 \\ \text{unique non-trivial involution} & |Q| = A_n, D_{2n+1}, E_6 \end{cases}$$

2. This determines an involution of $\overline{Q}$ by requiring $\eta(\alpha) : \eta(i) \to \eta(j)$ for $\alpha : i \to j$ in $\overline{Q}$.
3. Determines an involution of $k\overline{Q}$ preserving (ρ) and so gives an involution η of Π_Q.

Example

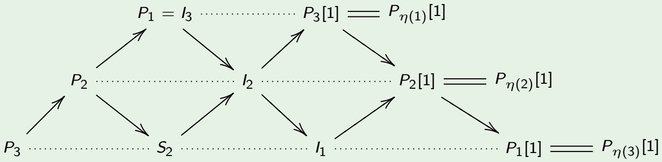


The Preprojective Algebra

The involution η is used to construct $\mathscr{D}^b(kQ)$ from mod kQ:

Example

$Q : 1 \to 2 \to 3$



The Homology of Γ_Q

Theorem (H.)

Suppose Q is Dynkin. Then there is an algebra isomorphism

$$H^* \Gamma_Q \cong \Pi_Q^\eta[u]$$

where $\Pi_Q^\eta[u]$ *is the* η-**twisted polynomial algebra***. Moreover, under this isomorphism, polynomial degree corresponds to weight.*

As a k-vector space $\Pi_Q^\eta[u] = \Pi_Q \otimes_k k[u]$; the multiplication is given by

$$(xu^p) \cdot (yu^q) = x\eta^p(y)u^{p+q}$$

for $x, y \in \Pi_Q$.

Remark on the Proof

The proof is given by showing both $H^*\Gamma_Q$ and $\Pi_Q^\eta[u]$ are isomorphic to

$$\bigoplus_{n\geq 0} H^* \mathscr{P}_{\mathrm{dg}}(kQ)(kQ, \tau^{-n} kQ)$$

where $\mathscr{P}_{\mathrm{dg}}(kQ)$ denotes the dg category of bounded projective complexes of kQ-modules with morphisms of arbitrary degree, and τ denotes the Auslander-Reiten translate.

The element $u \in \Pi_Q^\eta[u]$ comes from the evident map $kQ \to kQ[1]$.

The Homology of $\Gamma(Q)$

Corollary (Folklore?)

There is an isomorphism

$$H^*\Gamma_Q \cong \bigoplus_{n \geq 0} F^n kQ$$

in $\mathscr{D}^b(kQ)$ *where* $F = \tau^-[1]$.

Knowing $H^*\Gamma_Q$ is nice, but we really want Γ_Q. To recapture Γ_Q we need to know an A_∞-structure on $H^*\Gamma_Q$.

A_∞-Algebras

Definition

An A_∞-**algebra** is a k-module V together with "multiplications"

$$\mu_n : V^{\otimes n} \to V, \qquad n \geq 1$$

satisfying the relations

$$\sum_{\substack{n=p+q+r \\ q\geq 1, p,r\geq 0}} (-1)^{p+qr} \mu_{p+1+r} \circ \left(1^{\otimes p} \otimes \mu_q \otimes 1^{\otimes r}\right) = 0.$$

n=1: $\mu_1 \circ \mu_1 = 0$ i.e., (V, μ_1) is a chain complex

n=2: $\mu_2 \circ (1 \otimes \mu_1 + \mu_1 \otimes 1) = \mu_1 \circ \mu_2$
i.e., $\mu_2 : V \otimes V \to V$ is a chain map.

n=3: $\mu_2 \circ (\mu_2 \otimes 1) - \mu_2 \circ (1 \otimes \mu_2) = \mu_1 \circ \mu_3 + \mu_3 \circ d_{V^{\otimes 3}}$
i.e., μ_2 is associative up to a homotopy μ_3

n=4: ...

Examples

- Any ordinary associative algebra ($\mu_n = 0$ for $n \neq 2$); Conversely, for an A_∞-algebra V with $\mu_1 = 0$, (V, μ_2) is an associative algebra.
- Any differential graded algebra ($\mu_n = 0$ for $n \geq 2$)
- Any chain complex homotopy equivalent to an A_∞-algebra (not true for dgas!)

Remark

Two dgas (A, d_A, μ_A) and (B, d_B, μ_B) are quasi-isomorphic (as dgas) if and only if the A_∞-algebras $(A, d_A, \mu_A, 0, \dots)$ and $(B, d_B, \mu_B, 0, \dots)$ are quasi-isomorphic (as A_∞-algebras).

Kadeishvili's Theorem

Theorem (Kadeishvili)

Let A be a dga. Then there is a unique A_∞-algebra $(H^*A, \mu_1, \mu_2, \dots)$ so that:

- $\mu_1 = 0$
- μ_2 *is the usual multiplication*
- *the map $j : HA \to A$ given by choosing representative cycles is a quasi-isomorphism of A_∞-algebras.*

The A_∞-algebra H^*A above is called the **minimal model** of A. Kadeishvili's Theorem says dgas are determined (up to quiso) by their minimal models (up to A_∞-quiso).

The Minimal for Γ_Q

Recall there is an isomorphism $H^*\Gamma_Q \cong \Pi_Q^{\eta}[u]$.

Theorem (H.)

Suppose Q Dynkin and let $(H^*\Gamma_Q, 0, \mu_2, \mu_3, \dots)$ be the minimal model of Γ_Q.

1. *The maps μ_n are u-equivariant.*
2. *The element $u \in \mu_3\left(\Pi_Q^{\otimes 3}\right)$ and so $H^*\Gamma_Q$ is generated as an A_∞-algebra by Π_Q.*
3. *The higher multiplications $\mu_n = 0$ for $n > 3$.*

Remark on Proofs

Recall

$$H^*\Gamma_Q \cong \bigoplus_{n\geq 0} H^*\mathscr{P}_{\mathrm{dg}}(kQ)(kQ, \tau^{-n}kQ)$$

and u maps to $kQ \to kQ[1]$ under this isomorphism.

The category $\mathscr{P}_{\mathrm{dg}}(kQ)$ of projective complexes is a dg category so $H^*\mathscr{P}_{\mathrm{dg}}(kQ)$ is an A_∞-category. Transfers to A_∞-structure on $\bigoplus H^*\mathscr{P}_{\mathrm{dg}}(kQ)(kQ, \tau^{-n}kQ)$.

Proofs given by studying A_∞-structure on $H^*\mathscr{P}_{\mathrm{dg}}(kQ)$.

Thank You!