On the Homology of the Ginzburg Algebra

Stephen Hermes
Brandeis University, Waltham, MA

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Outline:

1. Ginzburg Algebra of a QP
2. Relation with the Preprojective Algebra
3. $A_\infty$-Algebras
4. Application to the Ginzburg Algebra
A quiver with potential (QP for short) is a pair \((Q, W)\) where \(Q\) is a quiver with

- no loops
- no 2-cycles

and \(W\) is a potential on \(Q\), i.e., an element of

\[
HH_0(kQ) = kQ/[kQ, kQ].
\]

Equivalently, \(W\) is a linear combination of cycles of \(Q\) considered up to cyclic equivalence.
The **Ginzburg algebra** $\Gamma_{(Q,W)}$ of a QP $(Q, W)$ is the dga constructed as follows. As a graded algebra, $\Gamma_{(Q,W)} = k\hat{Q}$ where $\hat{Q}$ is the quiver:

1. Start with $Q$ (in degree 0).
2. Add reversed arrows $\alpha^* : j \to i$ (degree $-1$) for each $\alpha : i \to j$ in $Q$.
3. Add loops $t_i$ (degree $-2$) for each vertex $i$ of $Q$.

**Example**

\[
\begin{array}{c}
1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3 \\
\end{array}
\]

$\sim$  

\[
\begin{array}{c}
1 \xrightarrow{\alpha^*} 2 \xrightarrow{\beta^*} 3 \\
\end{array}
\]
The Ginzburg Algebra

Equipped with a differential \( d \) determined by:

- \( d\alpha = 0 \) for \( \alpha \in Q_1 \)
- \( d\alpha^* = \partial_\alpha W \) for \( \alpha \in Q_1 \)

where \( \partial_\alpha : HH_0(kQ) \to kQ \) (the cyclic partial derivative) given by

\[
\partial_\alpha(w) = \sum_{w=u\alpha v} vu
\]

- \( dt_i = e_i \left( \sum_{\alpha \in Q_1} [\alpha, \alpha^*] \right) e_i \)
- \( ([x, y] = xy - yx) \).

Extend to all of \( \Gamma_{(Q,W)} \) by Leibniz law:

\[
d(xy) = d(x)y + (-1)^{|x|}xd(y).
\]

Example

\[
\begin{align*}
d\alpha &= d\beta = d\alpha^* = d\beta^* = 0 \\
dt_1 &= \alpha\alpha^*, dt_3 = -\beta^*\beta \\
dt_2 &= \beta\beta^* - \alpha^*\alpha
\end{align*}
\]
If $Q$ is acyclic (e.g. $Q$ Dynkin) $HH_0(kQ) = 0$; hence the only potential $Q$ admits is the trivial one $W = 0$. In this situation we write $\Gamma_Q = \Gamma_{(Q,0)}$.

For $Q$ acyclic $kQ = H^0\Gamma_Q$. But what about higher degrees?

**Definition**

Define the **weight** of a path $\gamma$ in $\hat{Q}$ to be the number of times $\gamma$ traverses a loop $t_i$.

Gives a weight grading on $\Gamma_Q$. Descends to a grading on $H^*\Gamma_Q$. Denote the weight $w$ component of $H^*\Gamma_Q$ by $H^w\Gamma_Q$. 
The Preprojective Algebra

Recall the **preprojective algebra** of $Q$ is the algebra $\Pi_Q = k\overline{Q}/(\rho)$ where

- $\overline{Q}$ is the subquiver of $\widehat{Q}$ consisting of arrows of weight 0.
- $\rho = \sum_{\alpha \in Q_1} [\alpha, \alpha^*]$.

**Example**

$$
\begin{align*}
1 & \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3 \\
\sim & \sim \\
1 & \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3
\end{align*}
$$

$\Pi_Q$ contains $kQ$ as a subalgebra. As a (right) $kQ$-module it splits into a direct sum of preprojective indecomposable modules with each isoclass represented exactly once.

$$
\Pi_Q = H^*_0 \Gamma_Q \subset H^* \Gamma_Q. \text{ This inclusion is proper in general.}
$$
The Preprojective Algebra

For $Q$ Dynkin, we have an (covariant) involution $\eta$ of $\Pi_Q$:

1. Let $\bar{\eta}$ be the involution of the underlying graph $|Q|$ of $Q$

   \[ \bar{\eta} = \begin{cases} 
   \text{identity map} & |Q| = D_{2n}, E_7, E_8 \\
   \text{unique non-trivial involution} & |Q| = A_n, D_{2n+1}, E_6
   \end{cases} \]

2. This determines an involution of $\bar{Q}$ by requiring $\eta(\alpha): \eta(i) \to \eta(j)$ for $\alpha : i \to j$ in $\bar{Q}$.

3. Determines an involution of $k\bar{Q}$ preserving $(\rho)$ and so gives an involution $\eta$ of $\Pi_Q$.

Example

\[
\begin{array}{c}
1 \\
3 \\
\end{array} \quad \begin{array}{c}
\rightarrow \\
\rightarrow \\
\end{array} \quad \begin{array}{c}
2 \\
\cdots \\
(2n + 1)
\end{array} \quad \begin{array}{c}
\eta \\
\rightarrow \\
\rightarrow \\
\end{array} \quad \begin{array}{c}
3 \\
1 \\
\end{array} \quad \begin{array}{c}
\rightarrow \\
\rightarrow \\
\end{array} \quad \begin{array}{c}
2 \\
\cdots \\
(2n + 1)
\end{array}
\]

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The involution $\eta$ is used to construct $\mathcal{D}^b(kQ)$ from mod $kQ$:

**Example**

$Q : 1 \rightarrow 2 \rightarrow 3$

[Diagram showing the relations between $P_1$, $P_2$, $P_3$, $l_1$, $l_2$, $l_3$, $S_2$, with arrows indicating the connections between them.]
The Homology of $\Gamma_Q$

Theorem (H.)

Suppose $Q$ is Dynkin. Then there is an algebra isomorphism

$$H^*\Gamma_Q \cong \Pi^\eta_Q[u]$$

where $\Pi^\eta_Q[u]$ is the $\eta$-twisted polynomial algebra. Moreover, under this isomorphism, polynomial degree corresponds to weight.

As a $k$-vector space $\Pi^\eta_Q[u] = \Pi_Q \otimes_k k[u]$; the multiplication is given by

$$(xu^p) \cdot (yu^q) = x\eta^p(y)u^{p+q}$$

for $x, y \in \Pi_Q$. 
Remark on the Proof

The proof is given by showing both $H^*\Gamma_Q$ and $\Pi^\eta_Q[u]$ are isomorphic to

$$\bigoplus_{n \geq 0} H^* \mathcal{P}_{dg}(kQ)(kQ, \tau^{-n}kQ)$$

where $\mathcal{P}_{dg}(kQ)$ denotes the dg category of bounded projective complexes of $kQ$-modules with morphisms of arbitrary degree, and $\tau$ denotes the Auslander-Reiten translate.

The element $u \in \Pi^\eta_Q[u]$ comes from the evident map $kQ \to kQ[1]$. 
Corollary (Folklore?)

There is an isomorphism

\[ H^* \Gamma_Q \cong \bigoplus_{n \geq 0} F^n kQ \]

in \( \mathcal{D}^b(kQ) \) where \( F = \tau^{-}[1] \).

Knowing \( H^* \Gamma_Q \) is nice, but we really want \( \Gamma_Q \). To recapture \( \Gamma_Q \) we need to know an \( A_\infty \)-structure on \( H^* \Gamma_Q \).
**Definition**

An $A_\infty$-algebra is a $k$-module $V$ together with “multiplications”

$$\mu_n : V^\otimes n \to V, \quad n \geq 1$$

satisfying the relations

$$\sum_{\substack{n=p+q+r \geq 1, p, r \geq 0 \atop q \geq 1}} (-1)^{p+qr} \mu_{p+1+r} \circ (1^\otimes p \otimes \mu_q \otimes 1^\otimes r) = 0.$$ 

**n=1:** $\mu_1 \circ \mu_1 = 0$ i.e., $(V, \mu_1)$ is a chain complex

**n=2:** $\mu_2 \circ (1 \otimes \mu_1 + \mu_1 \otimes 1) = \mu_1 \circ \mu_2$

i.e., $\mu_2 : V \otimes V \to V$ is a chain map.

**n=3:** $\mu_2 \circ (\mu_2 \otimes 1) - \mu_2 \circ (1 \otimes \mu_2) = \mu_1 \circ \mu_3 + \mu_3 \circ d_{V^\otimes 3}$

i.e., $\mu_2$ is associative up to a homotopy $\mu_3$

**n=4:** ...
Examples

- Any ordinary associative algebra ($\mu_n = 0$ for $n \neq 2$); Conversely, for an $A_\infty$-algebra $V$ with $\mu_1 = 0$, $(V, \mu_2)$ is an associative algebra.
- Any differential graded algebra ($\mu_n = 0$ for $n \geq 2$)
- Any chain complex homotopy equivalent to an $A_\infty$-algebra (not true for dgas!)

Remark

Two dgas $(A, d_A, \mu_A)$ and $(B, d_B, \mu_B)$ are quasi-isomorphic (as dgas) if and only if the $A_\infty$-algebras $(A, d_A, \mu_A, 0, \ldots)$ and $(B, d_B, \mu_B, 0, \ldots)$ are quasi-isomorphic (as $A_\infty$-algebras).
Kadeishvili’s Theorem

**Theorem (Kadeishvili)**

Let $A$ be a dga. Then there is a unique $A_\infty$-algebra $(H^*A, \mu_1, \mu_2, \ldots)$ so that:

- $\mu_1 = 0$
- $\mu_2$ is the usual multiplication
- the map $j : HA \to A$ given by choosing representative cycles is a quasi-isomorphism of $A_\infty$-algebras.

The $A_\infty$-algebra $H^*A$ above is called the **minimal model** of $A$. Kadeishvili’s Theorem says dgas are determined (up to quiso) by their minimal models (up to $A_\infty$-quiso).
The Minimal for $\Gamma_Q$

Recall there is an isomorphism $H^*\Gamma_Q \cong \Pi^\eta_Q[u]$.

**Theorem (H.)**

Suppose $Q$ Dynkin and let $(H^*\Gamma_Q, 0, \mu_2, \mu_3, \ldots)$ be the minimal model of $\Gamma_Q$.

1. The maps $\mu_n$ are $u$-equivariant.
2. The element $u \in \mu_3 \left( \Pi^\otimes_3 Q \right)$ and so $H^*\Gamma_Q$ is generated as an $A_\infty$-algebra by $\Pi_Q$.
3. The higher multiplications $\mu_n = 0$ for $n > 3$. 

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Remark on Proofs

Recall

\[ H^* \Gamma_Q \cong \bigoplus_{n \geq 0} H^* \mathcal{P}_{dg}(kQ)(kQ, \tau^{-n} kQ) \]

and \( u \) maps to \( kQ \to kQ[1] \) under this isomorphism.

The category \( \mathcal{P}_{dg}(kQ) \) of projective complexes is a dg category so \( H^* \mathcal{P}_{dg}(kQ) \) is an \( A_\infty \)-category. Transfers to \( A_\infty \)-structure on \( \bigoplus H^* \mathcal{P}_{dg}(kQ)(kQ, \tau^{-n} kQ) \).

Proofs given by studying \( A_\infty \)-structure on \( H^* \mathcal{P}_{dg}(kQ) \).
Thank You!